# THE INFRARED SINGULARITIES OF MASSLESS GAUGE THEORIES

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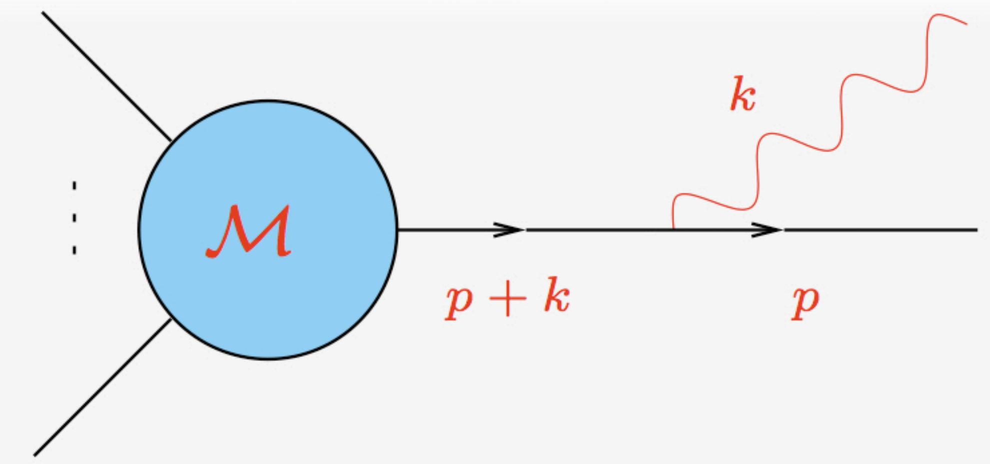
# Outline

- Infrared divergences to all orders
  - Theory
  - Practice
  - Tools
- The dipole formula (with E. Gardi)
- The high-energy limit (with V. Del Duca, C. Duhr, E. Gardi, C. White)
- Outlook

# ON INFRARED DIVERGENCES

# Textbook theory ...




$$\rightarrow \quad -ig\overline{u}(p)\not{\epsilon}(k)t_a \frac{i(\not{p}+\not{k})}{(p+k)^2 + i\varepsilon}\mathcal{M}\,,$$

Singularities arise when propagators go on shell.

$$2p \cdot k = 2p_0 k_0 (1 - \cos\theta_{pk}) = 0\,,$$
$$\rightarrow k_0 = 0 \ (IR); \quad \cos\theta_{pk} = 1\,.$$

➡ Emission is not suppressed at long distances.

➡ Isolated charged particles are not true asymptotic states of unbroken gauge theories.

- A serious problem: the S matrix does not exist in the usual Fock space.
- Possible solutions: construct finite transition probabilities (KLN theorem); construct better asymptotic states (coherent states).
- Long-distance singularities obey a pattern of exponentiation

$$\mathcal{M} = \mathcal{M}_0 \left[1 - \kappa\,\frac{\alpha}{\pi}\,\frac{1}{\epsilon} + \ldots\right] \ \Rightarrow \ \mathcal{M} = \mathcal{M}_0 \exp\left[-\kappa\,\frac{\alpha}{\pi}\,\frac{1}{\epsilon} + \ldots\right]$$

# ... and Practice

Just a formal issue in Quantum Field Theory? Are there practical applications?

- Higher order QCD calculations at colliders hinge upon cancellation of divergences between virtual corrections and real emission contributions.
  - Cancellation must be performed analytically before numerical integrations.
  - Need local counterterms for matrix elements in all singular regions.
  - State of the art: NLO multileg, NNLO for (some) color-singlet processes.

- Cancellations leave behind large logarithms: they must be resummed

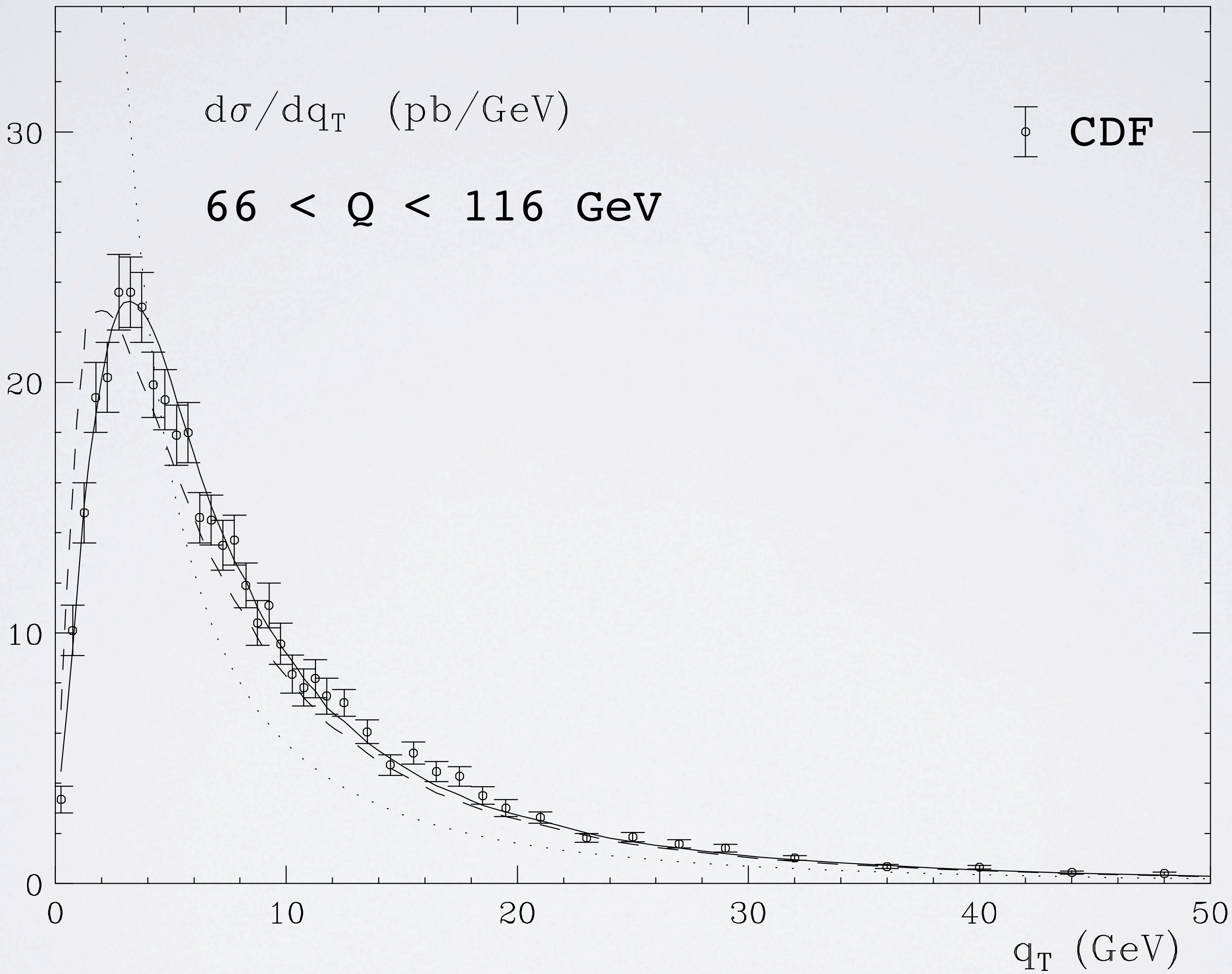
$$\underbrace{\frac{1}{\epsilon}}_{\text{virtual}} + \underbrace{(Q^2)^\epsilon \int_0^{m^2} \frac{dk^2}{(k^2)^{1+\epsilon}}}_{\text{real}} \qquad \Longrightarrow \qquad \ln(m^2/Q^2)$$

  - For inclusive observables: analytic resummation to high logarithmic accuracy.
  - For exclusive final states: parton shower event generators, (N)LL accuracy.

- Resummation probes the all-order structure of perturbation theory.
  - Power-suppressed corrections to QCD cross sections can be studied.
  - Links to the strong coupling regime can be established for SUSY gauge theories.

# Impact of resummation

Z-boson $q_T$ spectrum at Tevatron (A. Kulesza et al.)




CDF data on $Z$ production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).

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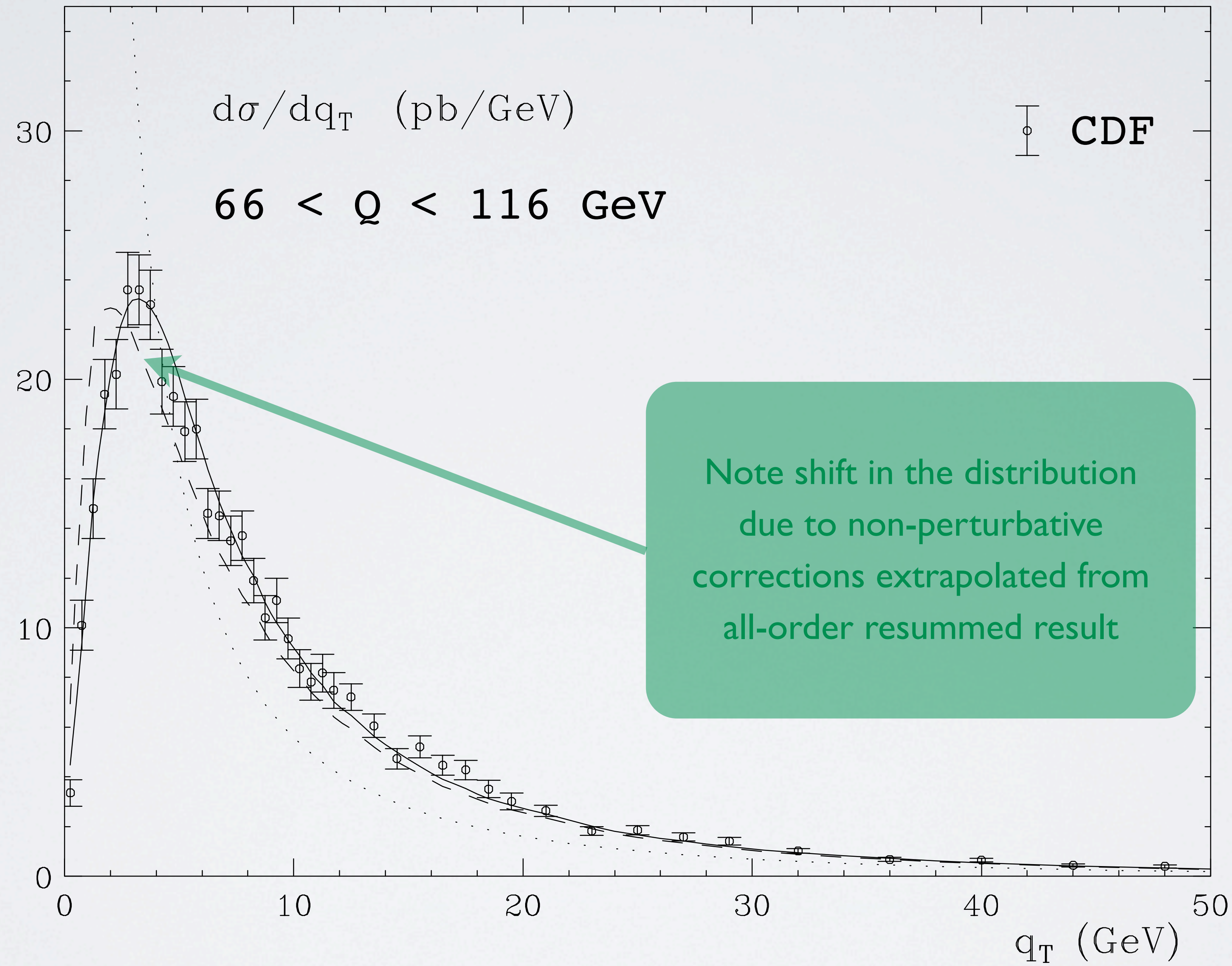


CDF data on $Z$ production compared with QCD predictions at fixed order (dotted), with resummation (dashed), and with the inclusion of power corrections (solid).

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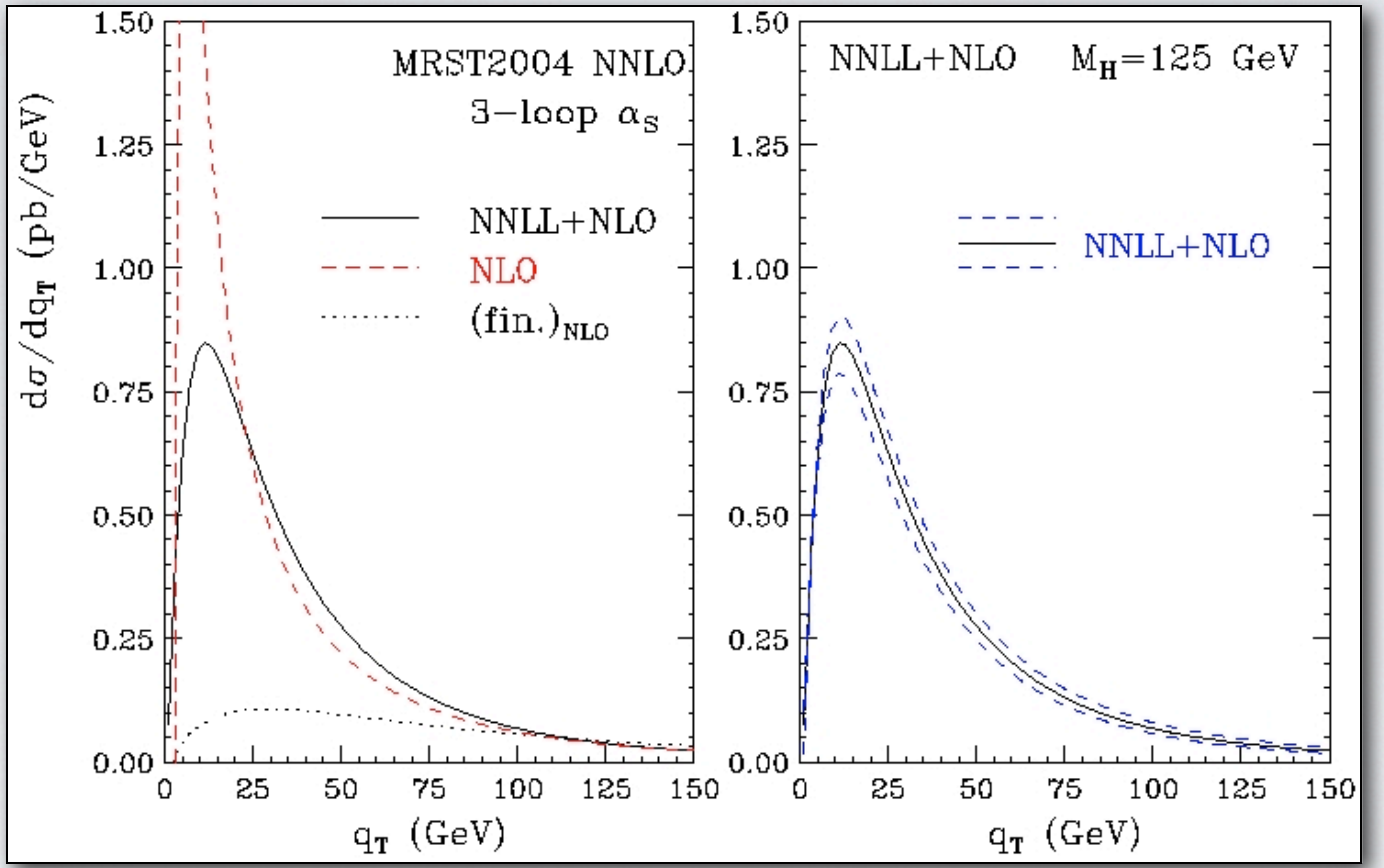
Predictions for the Higgs boson $q_T$ spectrum at LHC (M. Grazzini)




Predictions for the $q_T$ spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation, and theoretical uncertainty band of the resummed prediction.

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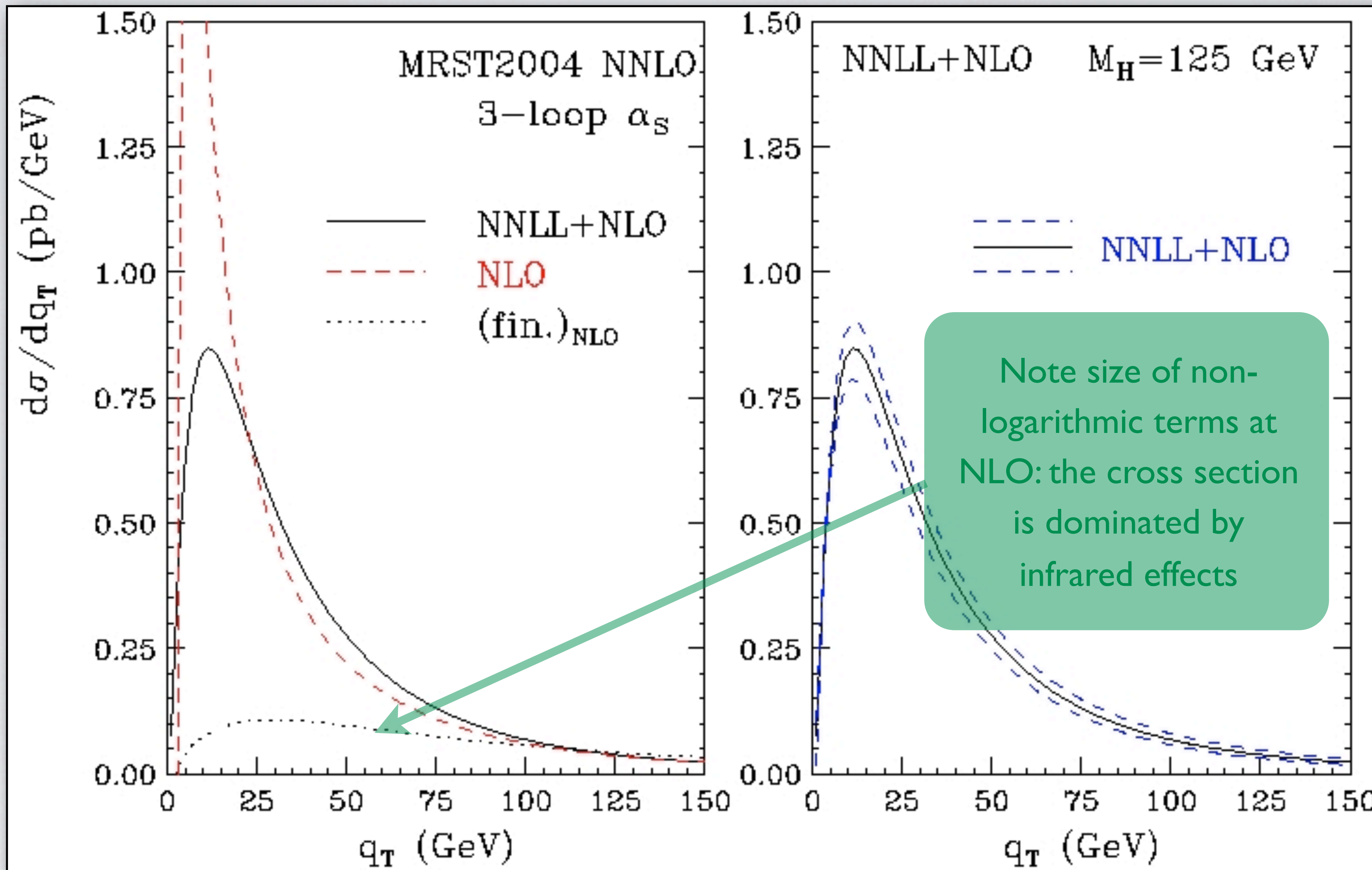


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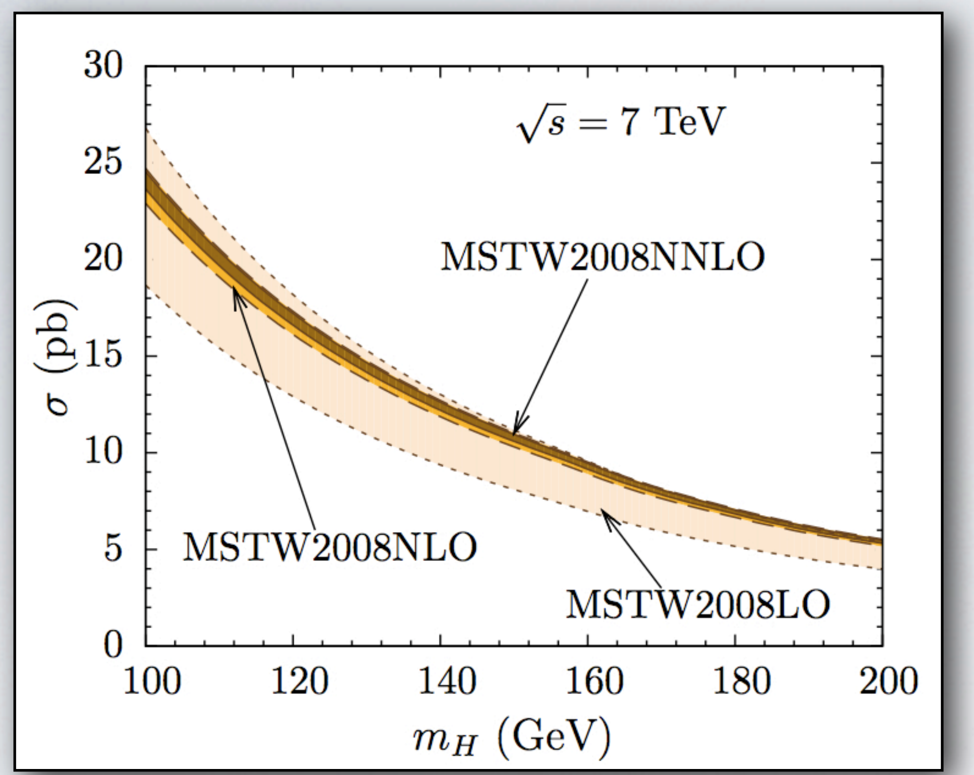
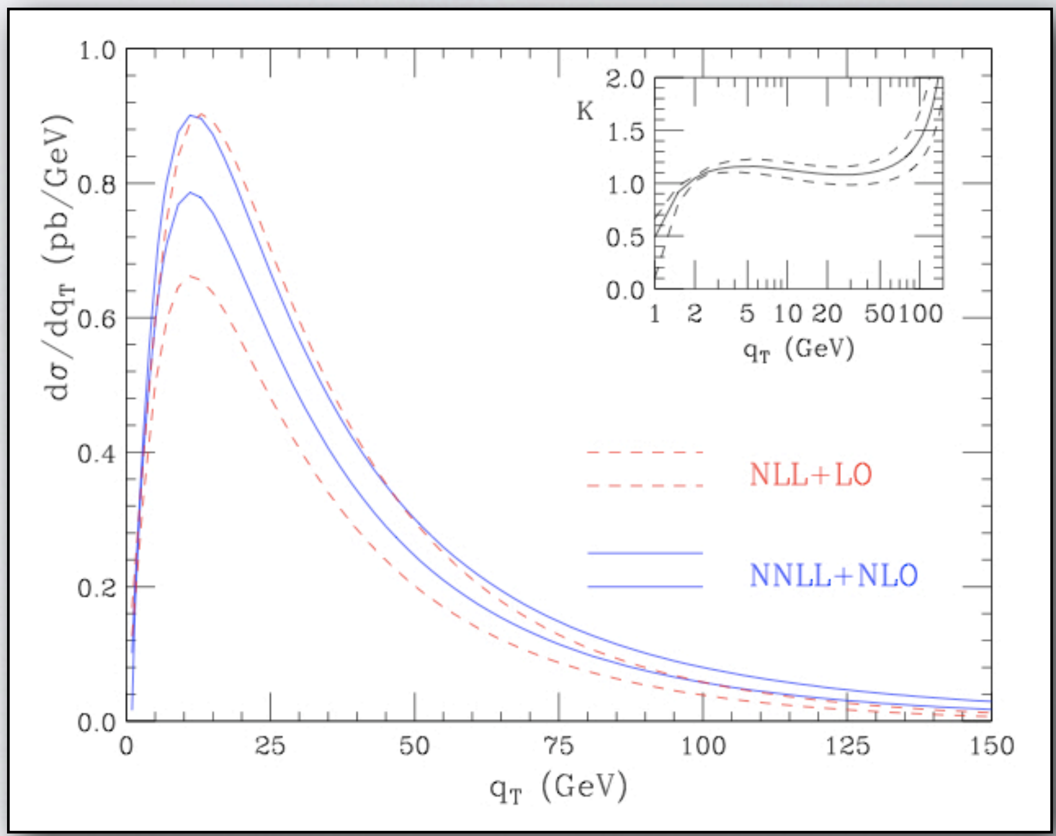
# The case of Higgs production




N³LL resummed cross section for Higgs production via gluon fusion at LHC

- The total cross section for gg->H is known to N³LL and NNLO, with NLO EW corrections.
  - One of the best-known observables in the SM.
  - A combined analysis (Ahrens et al. 11) gives a 3% (th) + 8% (pdf) + 1% ($m_q$) uncertainty.
  - Ongoing debate on theoretical and pdf uncertainty (Baglio et al. 11).

- The $p_T$ distribution for gg->H is known to NNLL and NNLO (M. Grazzini et al. '07, '10; T. Becher et al. '11).
  - Resummation reduces scale uncertainty.
  - A subtle polarization effect uncovered but not implemented yet (Catani, Grazzini, '10).
  - Three-loop coefficient recently computed: theoretically interesting, small impact (Becher, Neubert, '11).




NNLL resummed pT distribution for Higgs production via gluon fusion at LHC

# IR divergences and Jets

- Theoretically sound jet algorithms guarantee the cancellation of IR singularities to all orders (SISCone, (anti)-$k_T$, Cambridge-Aachen, ... ). Is this important?
- For Theory: unsafe algorithms give meaningless predictions beyond a given order.
- For Experiment: unsafe algorithms yield, event by event, a jet content that is unstable against the emission of a soft pion or a collinear decay.
- What does the divergence mean in practice?

$$\sigma = \sigma_0 \left(1 + c_1 \alpha_s + c_2 \alpha_s^2 + \ldots\right) \qquad \ldots \qquad c_2 = \infty\,!$$

$$\sigma = \sigma_0 \left(1 + c_1 \alpha_s + K \log\left(\frac{\Lambda}{Q}\right) \alpha_s^2 + \ldots\right) = \sigma_0 \left(1 + (c_1 + K)\alpha_s + \ldots\right) .$$

- Infrared sensitivity at $N^kLO$ destroys the predictivity of a $N^{k-1}LO$ calculation.
- Impact depends on the specific algorithm and observable.
  - The single-inclusive jet cross section is least affected: $\delta\sigma/\sigma < 5\%$ comparing SISCone and MidPoint Cone algorithms.
  - Multi-Jet cross sections are severely affected.
    - ➡ W + n-jets existing NLO predictions ($n = 2,3$) are not applicable to MidPoint Cone algorithms ... the processes are then unreliable as discovery channels ...
    - ➡ Studies of boosted heavy particles are only feasible with IR safe algorithms.

# TOOLS

# Dimensional regularization

Exponentiation of infrared poles requires solving d-dimensional evolution equations. The running coupling in d = 4 - 2 ε obeys

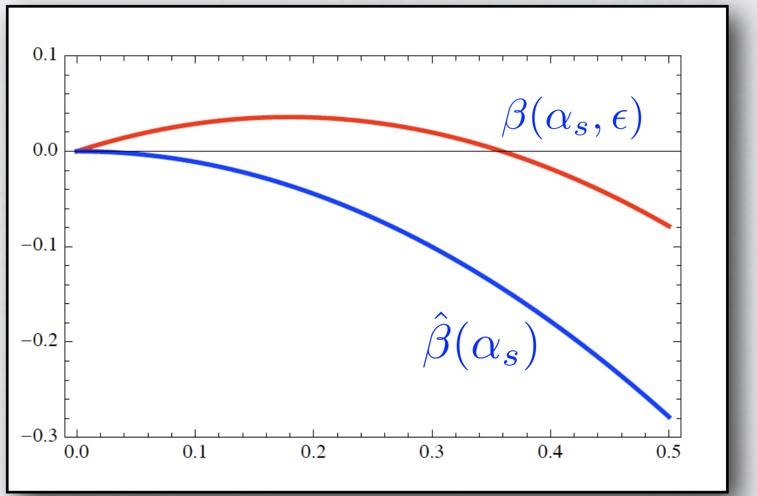
$$\mu \frac{\partial \overline{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \overline{\alpha}) = -2\,\epsilon\,\overline{\alpha} + \hat{\beta}(\overline{\alpha}) \quad , \quad \hat{\beta}(\overline{\alpha}) = -\frac{\overline{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\overline{\alpha}}{\pi}\right)^n \, .$$

The one-loop solution is

$$\overline{\alpha}\left(\mu^2, \epsilon\right) = \alpha_s(\mu_0^2) \left[ \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^{\epsilon}\right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} \, .$$

The β function develops an IR-free fixed point, so that the coupling vanishes at μ = 0 for fixed ε < 0. The Landau pole is at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0 \alpha_s(Q^2)}\right)^{-1/\epsilon} \, .$$




For negative ε the beta function develops a second zero, O(ε) from the origin.

- ➡ Integrations over the scale of the coupling can be analytically performed.
- ➡ All infrared and collinear poles arise by integration over the scale of the running coupling.

# Factorization: UV

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

Renormalization is a textbook example.

- Renormalization factorizes cutoff dependence.

$$G_0^{(n)}\left(p_i, \Lambda, g_0\right) = \prod_{i=1}^{n} Z_i^{1/2}\left(\Lambda/\mu, g(\mu)\right)\; G_R^{(n)}\left(p_i, \mu, g(\mu)\right)$$

- Factorization requires the introduction of an arbitrarily chosen scale $\mu$.

- Results must be independent of the arbitrary choice of $\mu$.

$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d \log G_R^{(n)}}{d \log \mu} = -\sum_{i=1}^{n} \gamma_i\left(g(\mu)\right) \; .$$

- The simple functional dependence of the factors is dictated by separation of variables.

- Proving factorization is the difficult step: it requires all-order diagrammatic analyses. Evolution equations follow automatically.

- Solving RG evolution resums logarithms of $Q^2/\mu^2$ into $\alpha_s(\mu^2)$.

# Factorization: Collinear

A textbook example is collinear factorization for DIS structure functions.

- Collinear factorization separates the dependence on the physical scale $Q^2$ from the dependence on collinear cutoffs (parton masses $m^2$). For Mellin moments one gets

$$\widetilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s\right) = \widetilde{C}\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s\right) \widetilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s\right) .$$

- Factorization requires the introduction of an arbitrarily chosen scale $\mu_F$. Results must be independent of the arbitrary choice of $\mu_F$.

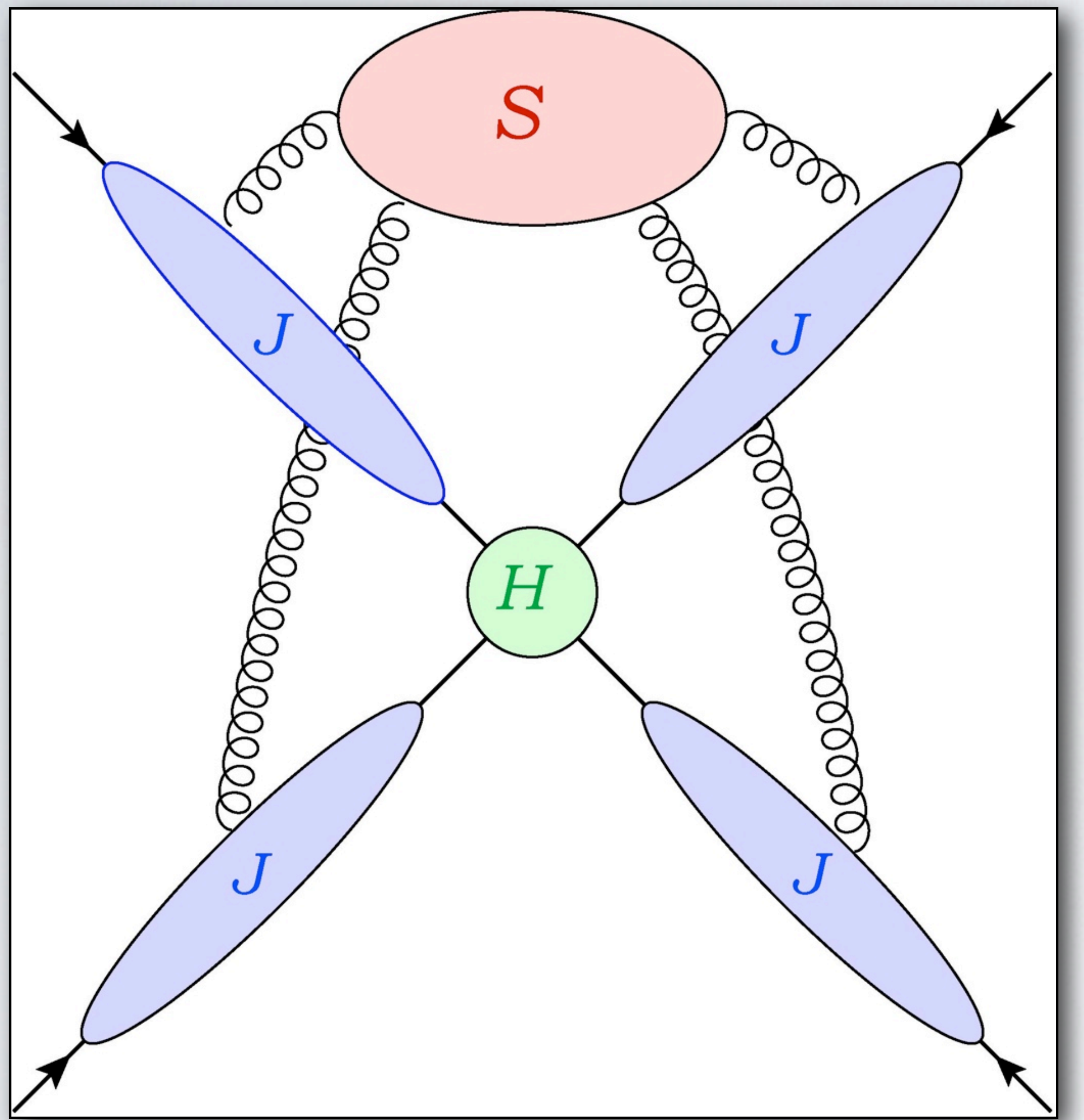
$$\frac{d\widetilde{F}_2}{d\mu_F} = 0 \quad \longrightarrow \quad \frac{d\log \widetilde{f}}{d\log \mu_F} = \gamma_N\left(\alpha_s\right) .$$

- The simple functional dependence of the factors is dictated by separation of variables.

- Proving factorization is the difficult step: it requires all-order diagrammatic analyses, or OPE. Evolution equations for parton distributions follow automatically.

- Solving Altarelli-Parisi evolution resums logarithms of $Q^2/\mu_F^2$ into evolved parton distributions (or fragmentation functions).

# Factorization: Sudakov



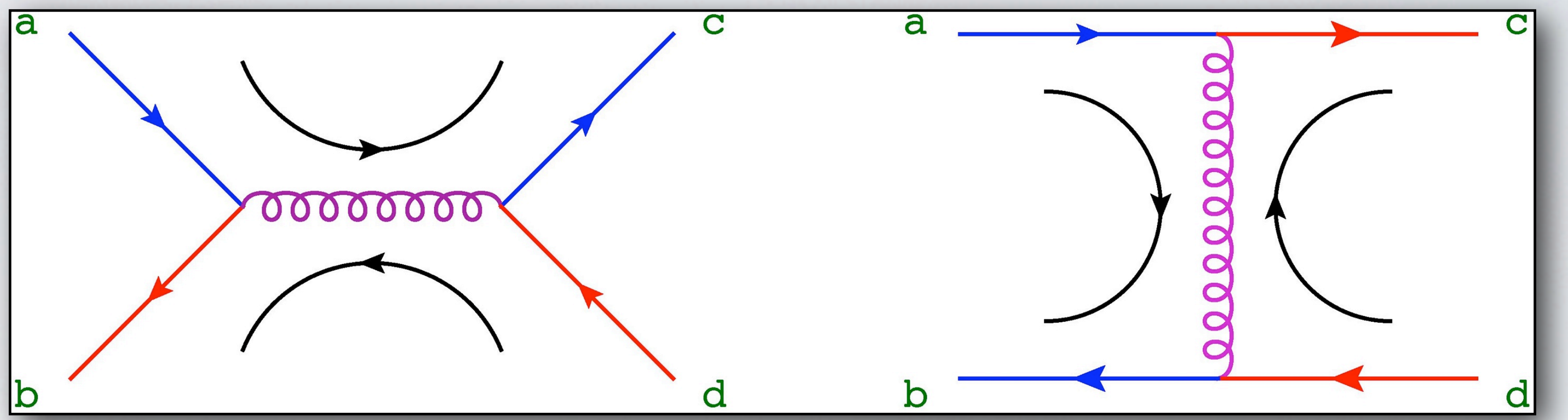

Leading integration regions in loop momentum space
for Sudakov factorization

- Sudakov logarithms are remainders of infrared and collinear divergences.
- Divergences arise in scattering amplitudes from leading regions in loop momentum space.
- Soft gluons factorize both form hard (easy) and from collinear (intricate) virtual exchanges.
- Jet functions J represent color singlet evolution of external hard partons.
- The soft function S is a matrix mixing the available color representations.
- In the planar limit soft exchanges are confined to wedges: S is proportional to the identity.
- In the planar limit S can be reabsorbed defining jets as square roots of elementary form factors.
- Beyond the planar limit S is determined by an anomalous dimension matrix $\Gamma_S$.

# Color flow

In order to understand the matrix structure of the soft function it is sufficient to consider the simple case of quark-antiquark scattering.

At tree level




Tree-level diagrams and color flows for quark-antiquark scattering

For this process only two color structures are possible. A basis in the space of available color tensors is

$$c^{(1)}_{abcd} = \delta_{ab}\delta_{cd} \, , \qquad c^{(2)}_{abcd} = \delta_{ac}\delta_{bd}$$

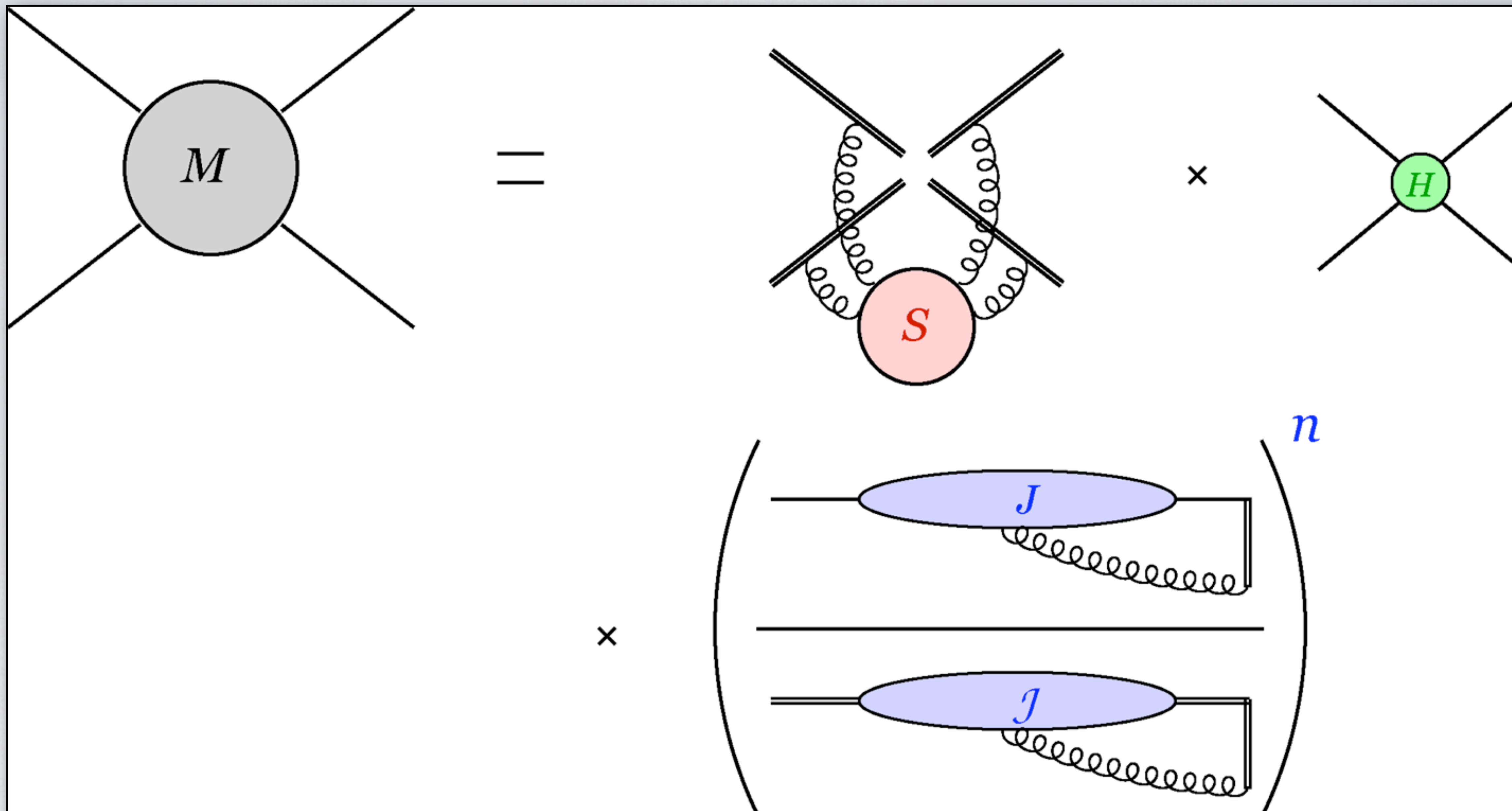
The matrix element is a vector in this space, and the Born cross section is

$$\mathcal{M}_{abcd} = \mathcal{M}_1 \, c^{(1)}_{abcd} + \mathcal{M}_2 \, c^{(2)}_{abcd} \longrightarrow \sum_{\text{color}} |\mathcal{M}|^2 = \sum_{J,L} \mathcal{M}_J \, \mathcal{M}^*_L \; \text{tr}\left[ c^{(J)}_{abcd} \left( c^{(L)}_{abcd} \right)^\dagger \right] \equiv \text{Tr}\Big[ HS \Big]_0$$

A virtual soft gluon will reshuffle color and mix the components of this vector

$$\text{QED}: \quad \mathcal{M}_{\text{div}} = S_{\text{div}} \mathcal{M}_{\text{Born}} \, ; \qquad \text{QCD}: \quad \left[ \mathcal{M}_{\text{div}} \right]_J = \left[ S_{\text{div}} \right]_{JL} \left[ \mathcal{M}_{\text{Born}} \right]_L$$

# Sudakov factorization: pictorial




A pictorial representation of Sudakov factorization for fixed-angle scattering amplitudes

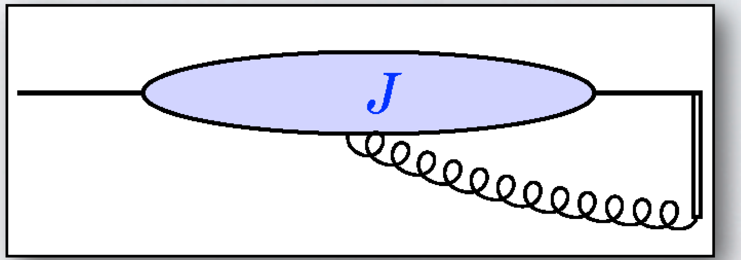
# Operator Definitions

The precise functional form of this graphical factorization is

$$\mathcal{M}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \times \prod_{i=1}^{n}\left[J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)\right],$$

Here we introduced dimensionless four-velocities $\beta_i^\mu = Q\, p_i^\mu$, $\beta_i^2 = 0$, and factorization vectors $n_i^\mu$, $n_i^2 \neq 0$ to define the jets,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 | \Phi_n(\infty, 0)\, \psi(0)\, | p \rangle .$$




where $\Phi_n$ is the Wilson line operator along the direction $n^\mu$,

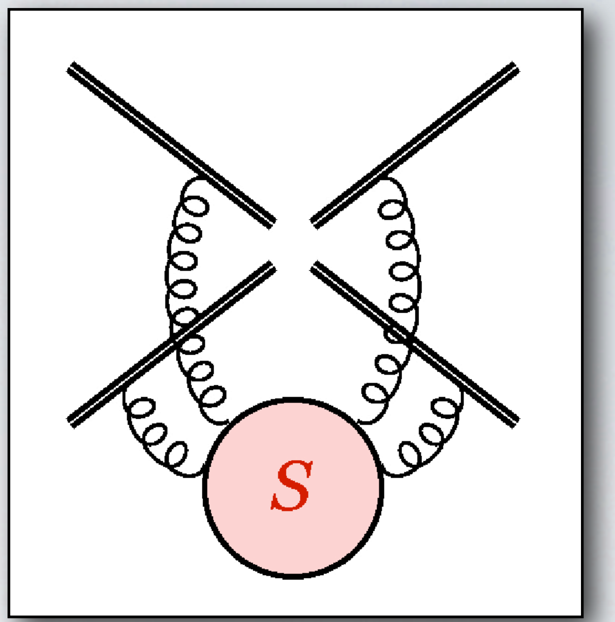
$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A(\lambda n)\right].$$

Note: Wilson lines represent fast particles, not recoiling against soft radiation

The vectors $n^\mu$:

- Ensure gauge invariance of the jets.
- Separate collinear gluons from wide-angle soft ones.
- Replace other hard partons with a collinear-safe absorber.

# Soft Matrices




The soft function S is a matrix, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$(c_L)_{\{a_k\}} \, \mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \epsilon\right) \; = \; \langle 0 | \prod_{k=1}^{n} \left[ \Phi_{\beta_k} \left(\infty, 0\right) \right]_{a_k}^{\;\;b_k} |0\rangle \; (c_K)_{\{b_k\}} \; ,$$

The soft function S obeys a matrix RG evolution equation

$$\mu \, \frac{d}{d\mu} \mathcal{S}_{LK} \left(\beta_i \cdot \beta_j, \epsilon\right) \; = \; - \, \mathcal{S}_{LJ} \left(\beta_i \cdot \beta_j, \epsilon\right) \, \Gamma^{\mathcal{S}}_{JK} \left(\beta_i \cdot \beta_j, \epsilon\right)$$

NOTE: $\Gamma^S$ is singular due to overlapping UV and collinear poles.

S is a pure counterterm. In dimensional regularization, using $\alpha_s(\mu^2 = 0, \varepsilon < 0) = 0$ ,

$$\mathcal{S} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) \; = \; P \, \exp \left[ - \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\xi^2, \epsilon), \epsilon\right) \right] \, .$$

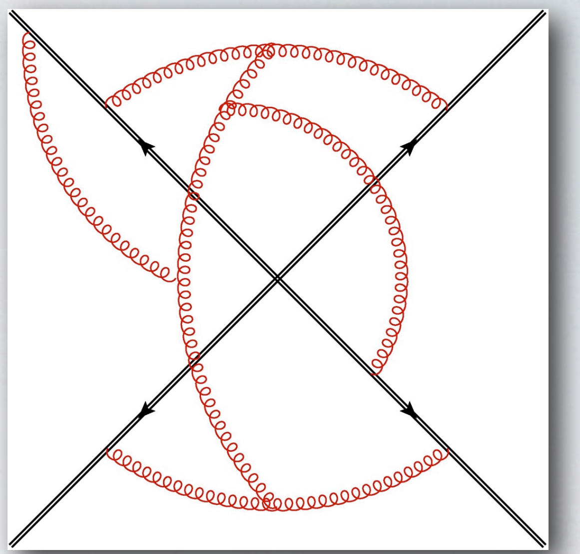
The determination of the soft anomalous dimension matrix $\Gamma^S$ is the keystone of the resummation program for multiparton amplitudes and cross sections.

- It governs the interplay of color exchange with kinematics in multiparton processes.
- It is the only source of multiparton correlations for singular contributions.
- Collinear effects are `color singlet' and can be extracted from two-parton scatterings.

# THE DIPOLE FORMULA

# Surprising Simplicity

- The matrix $\Gamma_S$ can be computed from the UV poles of S.
- Computations can be performed directly for the exponent: the relevant diagrams are called "webs".
- $\Gamma_S$ appears highly complex at high orders.
- g-loop webs directly correlate color and kinematics of up to g+1 Wilson lines: color multipoles are expected



A web contributing to the soft anomalous dimension matrix

The two-loop calculation (Aybat, Dixon, Sterman, 06) leads to a surprising result: for any number of external massless partons

$$\Gamma_{\mathcal{S}}^{(2)} = \frac{\kappa}{2}\,\Gamma_{\mathcal{S}}^{(1)} \qquad \kappa = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F C_F \,.$$

➡ No new kinematic dependence; no new matrix structure: only color dipoles survive

➡ $\kappa$ is the two-loop coefficient of the cusp anomalous dimension, $\gamma_K(\alpha_s)$, rescaled by the appropriate quadratic Casimir eigenvalue,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)} \left[ 2\,\frac{\alpha_s}{\pi} + \kappa \left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}\left(\alpha_s^3\right) \right] .$$

Note: the cusp (Korchemsky et al., 86-89) governs IR singularities for two-parton processes.

# The Dipole Formula

- The two-loop result led to an all-order understanding. For massless partons, the soft matrix obeys a set of exact equations that correlate color exchange with kinematics.
- The simplest solution to these equations is a sum over color dipoles (Becher, Neubert; Gardi, LM, 09). It leads to an ansatz for the all-order singularity structure of all multiparton fixed-angle massless scattering amplitudes: the dipole formula.
- All soft and collinear singularities can be collected in a multiplicative operator Z

$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = Z\left(\frac{p_i}{\mu_f}, \alpha_s(\mu_f^2), \epsilon\right) \mathcal{H}\left(\frac{p_i}{\mu}, \frac{\mu_f}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

- Z contains both soft singularities from S, and collinear ones from the jet functions. It must satisfy its own matrix RG equation

$$\frac{d}{d\ln\mu} Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = - Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) .$$

- The finite matrix Γ inherits the dipole structure from the soft matrix. It reads

$$\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{j\neq i}\ln\left(\frac{-2\,p_i\cdot p_j}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\mu^2)\right) .$$

- The color generators $\mathbf{T}_i$ act as gluon insertion operators on the hard amplitude

# Features of the dipole formula

- All known results for IR divergences of massless gauge theory amplitudes are recovered.
- The absence of multiparton correlations implies remarkable diagrammatic cancellations.
- The color matrix structure is fixed at one loop: path-ordering is not needed.
- All divergences are determined by a handful of anomalous dimensions.
- The cusp anomalous dimension plays a very special role: a universal IR coupling.
- The planar solution is naturally generalized to the non-planar case.

Can this be the definitive answer for IR divergences in massless non-abelian gauge theories?

- There are precisely two sources of possible corrections.
  - Quadrupole correlations may enter starting at three loops: they must be tightly constrained functions of conformal cross ratios of parton momenta.

$$\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) + \Delta\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) \, , \qquad \rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_k \, p_j \cdot p_l}$$

  - The cusp anomalous dimension may violate Casimir scaling beyond three loops.

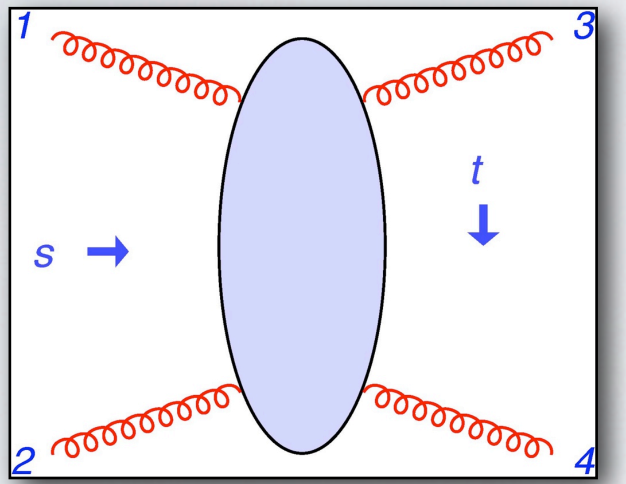
$$\gamma_K^{(i)}(\alpha_s) = C_i \, \widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)$$

  - The functional form of $\Delta$ is further constrained by: collinear limits, Bose symmetry and transcendentality bounds (Becher, Neubert; Dixon, Gardi, LM, 09).
  - Four-loop analysis indicates Casimir scaling holds (Becher, Neubert, Ahrens, Vernazza).

# THE HIGH-ENERGY LIMIT

# Reggeization

- Studies of the high-energy limit predate the modern era of quantum field theory.
- In the t/s → 0 limit particles exchanged in the t-channel (may) `Reggeize'.




Gluon-gluon scattering: the t-channel gluon Reggeizes

- Large logarithms of s/t are generated by a simple replacement of the t-channel propagator,

$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

- The Regge trajectory has a perturbative expansion, with IR divergent coefficients

$$\alpha(t) = \frac{\alpha_s(-t,\epsilon)}{4\pi}\,\alpha^{(1)} + \left(\frac{\alpha_s(-t,\epsilon)}{4\pi}\right)^2 \alpha^{(2)} + \mathcal{O}\left(\alpha_s^3\right)$$

- The gluon has been shown to Reggeize at NLL, and the two-loop Regge trajectory is known.
- For example, for gluon-gluon scattering the matrix element obeys Regge factorization

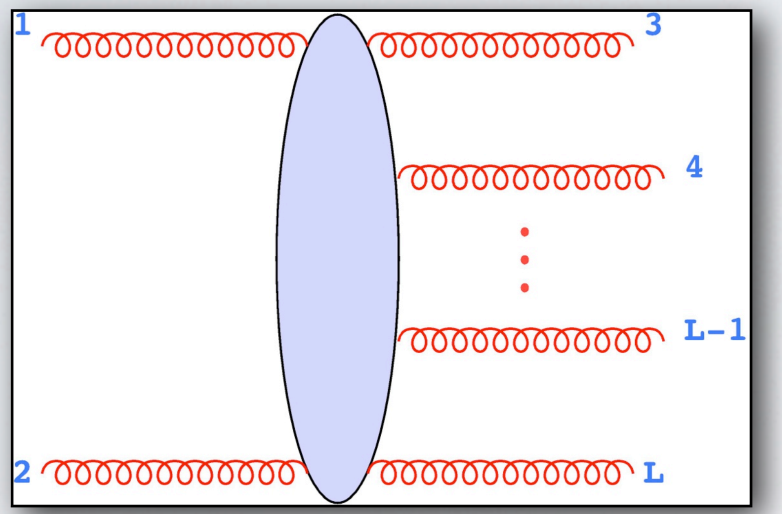
$$\mathcal{M}^{gg\to gg}_{a_1a_2a_3a_4}(s,t) = 2\,g_s^2\,\frac{s}{t}\left[(T^b)_{a_1a_3}C_{\lambda_1\lambda_3}(k_1,k_3)\right]\left(\frac{s}{-t}\right)^{\alpha(t)}\left[(T_b)_{a_2a_4}C_{\lambda_2\lambda_4}(k_2,k_4)\right]$$

with the perturbative coefficients

$$\alpha^{(1)} = C_A\,\frac{\widehat{\gamma}_K^{(1)}}{\epsilon} = C_A\,\frac{2}{\epsilon} \qquad \alpha^{(2)} = C_A\left[-\frac{b_0}{\epsilon^2} + \widehat{\gamma}_K^{(2)}\,\frac{2}{\epsilon} + C_A\left(\frac{404}{27} - 2\zeta_3\right) + n_f\left(-\frac{56}{27}\right)\right]$$

# Multi-Regge kinematics

- Reggeization follows form the dominance of t-channel ladder diagrams as t/s → 0.
- By unitarity, multi-gluon emission must similarly simplify in the high-energy limit.
- Regge factorization extends to multi-particle emission in \`Multi-Regge' kinematics.




Multi-gluon emission and Multi-Regge kinematics

$$y_3 \gg y_4 \gg \ldots \gg y_L \,, \qquad |k_i^\perp| \simeq |k_j^\perp| \,, \qquad \forall i, j$$

$$-s \equiv -s_{12} \simeq |k_3^\perp| \, |k_L^\perp| \, \mathrm{e}^{y_3 - y_L} \, \mathrm{e}^{\mathrm{i}\pi}$$

$$-s_{ij} \simeq |k_i^\perp| \, |k_j^\perp| \, \mathrm{e}^{y_i - y_j} \, \mathrm{e}^{\mathrm{i}\pi} \,, \quad 3 \leq i < j \leq L$$

- Large logarithms of $s/t_i$ are generated by the Reggeization of t-channel propagators, as

$$\mathcal{M}^{gg \to (L-2)g}_{a_1 \ldots a_L} = 2 g_s^3 \, s \left[ (T^b)_{a_1 a_3} \, C_{\lambda_1 \lambda_3}(k_1, k_3) \right] \left[ \frac{1}{t_1} \left( \frac{s_{34}}{-t_1} \right)^{\alpha(t_1)} \right]$$
$$\times \left[ (T^{a_4})_{bc} \, V_{\lambda_4}(q_1, q_2) \right] \left[ \frac{1}{t_2} \left( \frac{s_{45}}{-t_2} \right)^{\alpha(t_2)} \right] \cdots \left[ (T^c)_{a_2 a_L} \, C_{\lambda_2 \lambda_L}(k_2, k_L) \right]$$

- The impact factors C and the Lipatov vertices V are universal and independent of s.

# The dipole formula at high energy

- Introducing Mandelstam color operators, and using color and momentum conservation

$$\begin{aligned} \mathbf{T}_s &= \mathbf{T}_1 + \mathbf{T}_2 = -(\mathbf{T}_3 + \mathbf{T}_4) \,, \\ \mathbf{T}_t &= \mathbf{T}_1 + \mathbf{T}_3 = -(\mathbf{T}_2 + \mathbf{T}_4) \,, \\ \mathbf{T}_u &= \mathbf{T}_1 + \mathbf{T}_4 = -(\mathbf{T}_2 + \mathbf{T}_3) \end{aligned}$$

$$s + t + u = 0$$

$$\mathbf{T}_s^2 + \mathbf{T}_t^2 + \mathbf{T}_u^2 = \sum_{i=1}^{4} C_i$$

it is easy to see that the infrared dipole operator Z factorizes in the high-energy limit

$$Z\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) Z_{\mathbf{1}}\left(\frac{t}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)$$

- The operator $Z_1$ is s-independent and proportional to the unit matrix in color space.
- Color dependence and s dependence are collected in the factor

$$\widetilde{Z}\left(\frac{s}{t}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\ln\left(\frac{s}{-t}\right)\mathbf{T}_t^2 + \mathrm{i}\pi\,\mathbf{T}_s^2\right]\right\} ,$$

where the coupling dependence is (once again!) completely determined by the cusp anomalous dimension and by the β function, through the function (Korchemsky 94-96)

$$K\Big(\alpha_s(\mu^2), \epsilon\Big) \equiv -\frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)$$

- The simple structure of the high-energy operator governs Reggeization and its breaking.

# Reggeization of leading logarithms

- At leading logarithmic accuracy, the (imaginary) s-channel contribution can be dropped, and the dipole operator becomes diagonal in a t-channel basis.

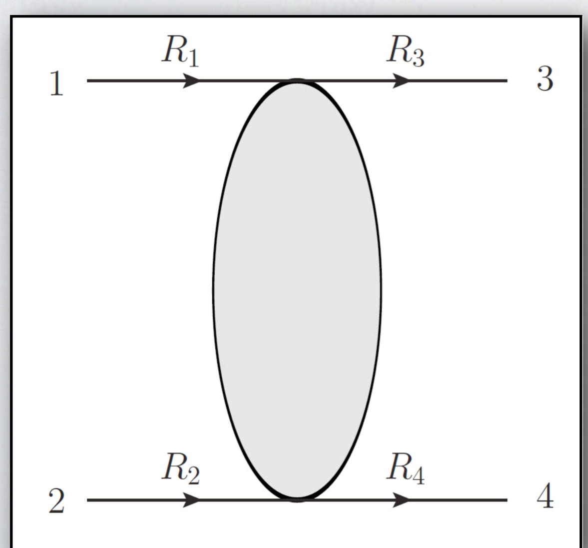
$$\mathcal{M}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{K\left(\alpha_s(\mu^2), \epsilon\right) \ln\left(\frac{s}{-t}\right) \mathbf{T}_t^2\right\} Z_{\mathbf{1}}\, \mathcal{H}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- If, at LO and at leading power in t/s, the scattering is dominated by t-channel exchange, then the hard function is an eigenstate of the color operator $\mathbf{T}_t^2$

$$\mathbf{T}_t^2\, \mathcal{H}^{gg\to gg} \xrightarrow{|t/s|\to 0} C_t\, \mathcal{H}_t^{gg\to gg}$$

- Leading-logarithmic Reggeization for arbitrary t-channel color representations follows

$$\mathcal{M}^{gg\to gg} = \left(\frac{s}{-t}\right)^{C_A\, K\left(\alpha_s(\mu^2), \epsilon\right)} Z_{\mathbf{1}}\, \mathcal{H}_t^{gg\to gg}$$




Scattering for generic color exchange

- The LL Regge trajectory is universal and obeys Casimir scaling.
- Scattering of arbitrary color representations can be analyzed Example: let 1 and 2 be antiquarks, 4 a gluon and 3 a sextet; use

$$\overline{\mathbf{3}} \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{15} \qquad \overline{\mathbf{3}} \otimes \mathbf{8}_a = \overline{\mathbf{3}} \oplus \mathbf{6} \oplus \overline{\mathbf{15}}$$

LL Reggeization of the $\mathbf{3}$ and $\mathbf{15}$ t-channel exchanges follows.

# Beyond leading logarithms

- The high-energy infrared operator can be systematically expanded beyond LL, using the Baker-Campbell-Hausdorff formula. At NLL one finds a series of commutators

$$\widetilde{Z}\left(\frac{s}{t},\alpha_s,\epsilon\right)\Big|_{\rm NLL} = \left(\frac{s}{-t}\right)^{K(\alpha_s,\epsilon)\,\mathbf{T}_t^2}\Bigg\{1+{\rm i}\,\pi K\left(\alpha_s,\epsilon\right)\Bigg[\mathbf{T}_s^2-\frac{K\left(\alpha_s,\epsilon\right)}{2!}\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right] + \frac{K^2\left(\alpha_s,\epsilon\right)}{3!}\ln^2\left(\frac{s}{-t}\right)\left[\mathbf{T}_t^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]+\ldots\Bigg]\Bigg\}$$

- The real part of the amplitude Reggeizes also at NLL for arbitrary t-channel exchanges.
- At NNLL Reggeization generically breaks down also for the real part of the amplitude.
  - At two loops, terms that are non-logarithmic and non-diagonal in a t-channel basis arise

$$\mathcal{E}_0\left(\alpha_s,\epsilon\right) \equiv -\frac{1}{2}\pi^2\,K^2\left(\alpha_s,\epsilon\right)\,\left(\mathbf{T}_s^2\right)^2$$

  - At three loops, the first Reggeization-breaking logarithms of s/t arise, generated by

$$\mathcal{E}_1\left(\frac{s}{t},\alpha_s,\epsilon\right) \equiv -\frac{\pi^2}{3}\,K^3(\alpha_s,\epsilon)\,\ln\left(\frac{s}{-t}\right)\left[\mathbf{T}_s^2,\left[\mathbf{T}_t^2,\mathbf{T}_s^2\right]\right]$$

- **NOTE**
  - In the planar limit ($N_C \to \infty$) all commutators vanish and Reggeization holds also beyond NLL (as perhaps expected from string theory).
  - Possible quadrupole corrections to the dipole formula cannot come to the rescue.

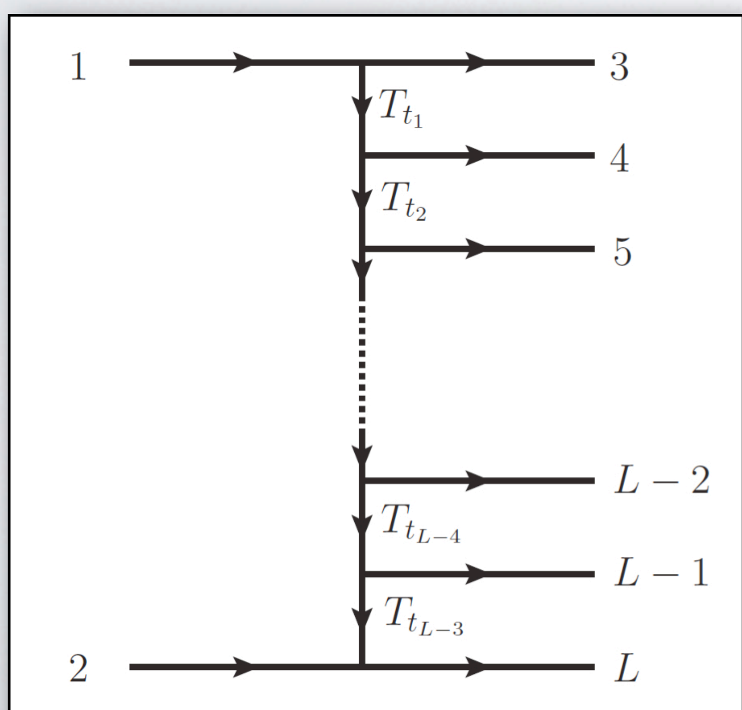
# Multi-Regge kinematics

- The dipole formula applies for any number of particles: we expect similar simplifications in Multi-Regge kinematics, and similar results concerning Reggeization.
- Indeed, one can prove recursively that the dipole operator Z factorizes in MR kinematics, as

$$Z\left(\frac{p_l}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \epsilon\Big)\; Z_{\mathbf{1}}^{\rm MR}\left(\frac{|k_i^\perp|}{\mu}, \alpha_s(\mu^2), \epsilon\right)$$

- The Multi-Regge high-energy operator has again a simple structure.

$$\widetilde{Z}^{\rm MR}\Big(\Delta y_k, \alpha_s(\mu^2), \epsilon\Big) = \exp\left\{K\Big(\alpha_s(\mu^2), \epsilon\Big)\left[\sum_{k=3}^{L-1} \mathbf{T}_{t_{k-2}}^2\, \Delta y_k + {\rm i}\pi \mathbf{T}_s^2\right]\right\}$$



Color structure in Multi-Regge kinematics

- We have defined the t-channel color operators

$$\mathbf{T}_{t_k} = \mathbf{T}_1 + \sum_{p=1}^{k} \mathbf{T}_{p+2}$$

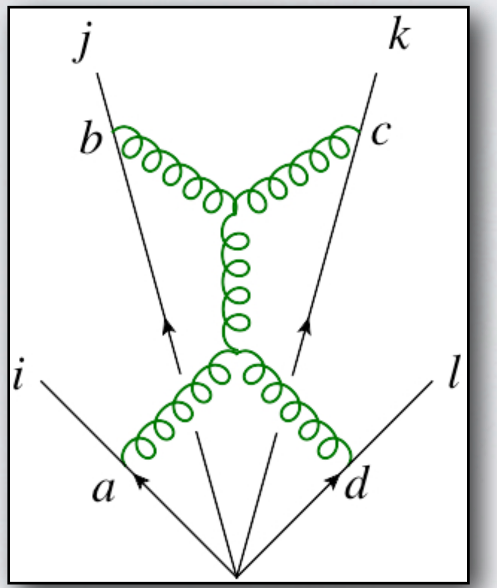
- A t-channel basis of common eigenstates of $\mathbf{T}_{t_k}$ can be constructed using Clebsch-Gordan coefficients.
- The operators $\mathbf{T}_{t_k}$ thus commute, and each color representation contributing to the hard function in the high-energy limit Reggeizes separately at LL.

# Constraining quadrupoles

- Known results on the high-energy limit of QCD amplitudes imply new constraints on quadrupole corrections to the dipole formula at three loops and beyond.
- Previous analyses using collinear constraint, Bose symmetry and transcendentality bounds could not exclude a class of correction, including for example

$$\Delta^{(212)}(\rho_{ijkl}, \alpha_s) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \left[ f^{ade} f^{cbe} L_{1234}^2 \left( L_{1423} L_{1342}^2 + L_{1423}^2 L_{1342} \right) + \text{cycl.} \right]$$

where $L_{ijkl} = \log(\rho_{ijkl})$.



A three-loop diagram for Δ

- In the high-energy limit one finds (with $L = \log|s/t|$ )

$$\rho_{1234} \equiv \frac{(-s_{12})(-s_{34})}{(-s_{13})(-s_{24})} = \left(\frac{s}{-t}\right)^2 e^{-2i\pi}; \qquad L_{1234} = 2(L - i\pi);$$

$$\rho_{1342} \equiv \frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} = \left(\frac{-t}{s+t}\right)^2; \qquad L_{1342} \simeq -2L;$$

$$\rho_{1423} \equiv \frac{(-s_{14})(-s_{23})}{(-s_{12})(-s_{34})} = \left(\frac{s+t}{s}\right)^2 e^{2i\pi}; \qquad L_{1423} \simeq 2i\pi,$$

- Previously admissible corrections display superleading high-energy logarithms at three loops.

$$\Delta^{(212)}(\rho_{ijkl}, \alpha_s)) = \left(\frac{\alpha_s}{\pi}\right)^3 \mathbf{T}_1^a \mathbf{T}_2^b \mathbf{T}_3^c \mathbf{T}_4^d \, 32\, i\, \pi \left[ \left(-L^4 - i\pi L^3 - \pi^2 L^2 - i\pi^3 L\right) f^{ade} f^{cbe} + \ldots \right]$$

- No known explicit example of admissible quadrupole correction survives. A complete proof is still lacking: linear combinations might restore the proper Regge behavior.

# OUTLOOK

# Summary

- After 7.5 $10^2$ years, soft and collinear singularities in gauge theories amplitudes are still a fertile field of study. A definitive solution may be at hand.
  - ✓ We are probing the all-order structure of the nonabelian exponent.
  - ✓ All-order results constrain, test and complement fixed-order calculations.
  - ✓ Understanding singularities has phenomenological applications through resummation.
- Factorization theorems ⇒ Evolution equations ⇒ Exponentiation.
  - ✓ Sudakov factorization ⇒ soft-gluon resummation.
  - ✓ Multiparton processes require anomalous dimension matrices.
- A simple dipole formula may encode all infrared singularites for any massless gauge theory, a natural generalization of the planar limit. The study of possible corrections to the dipole formula is under way.
- The high-energy limit of the dipole formula provides insights into Reggeization and beyond, at least for divergent contributions to the amplitude.
- Leading logarithmic Reggeization is proved for generic color representations exchanged in the t channel, and for any number of partons in Multi-Regge kinematics.
- Regge factorization generically breaks down at NNLL, with computable corrections.
- The high-energy limit further constrains quadrupole corrections to the dipole formula: no known examples survive.

$$\mathcal{M} = \mathcal{M}_0 \left[ 1 + \kappa \frac{\alpha_s}{\pi} \frac{1}{\epsilon} + \ldots \right]$$

CATASTROPHE?

$$\Gamma_{\rm dip}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{j\neq i}\ln\left(\frac{-2\,p_i\cdot p_j}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s(\mu^2)\right)\,.$$

SOLUTION?

CATASTROPHE !

SOLUTION?

# THANK YOU!