

# PROGRESS ON THE FIELD THEORY LIMIT OF PERTURBATIVE MULTI-LOOP STRING AMPLITUDES

Lorenzo Magnea

University of Torino - INFN Torino

Scattering Amplitudes, ECT\* Trento, 16/07/12



# Outline

- On string amplitudes
- One- and two-loop examples
- Effective actions in background fields
- Picking diagrams from strings
- Results and outlook

Work in collaboration with R. Russo (QMUL) and S. Sciuto (Torino)

# ON STRING AMPLITUDES



# ON STRING AMPLITUDES

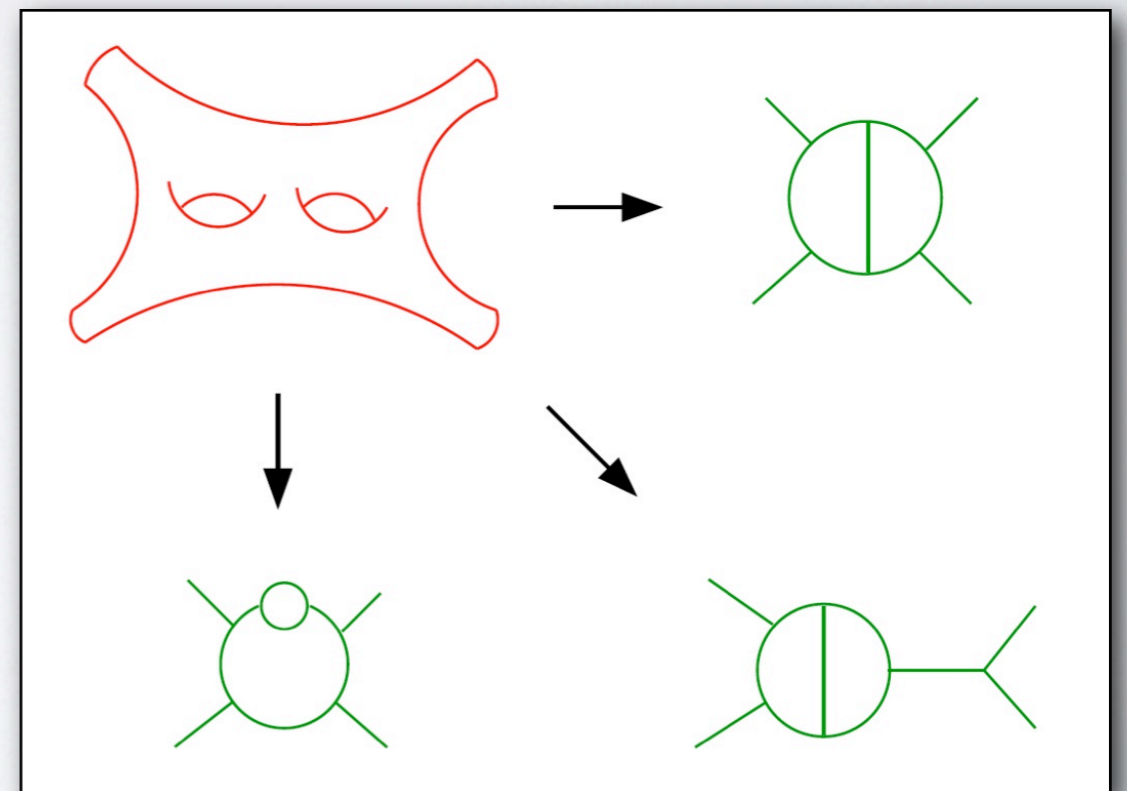


“Strings is a mythological story about the son of a king ...”

# Features of string amplitudes

- **String theory** expresses on-shell **scattering amplitudes** of a  **$d$ -dimensional interacting** field theory in terms of **correlation functions** of operators in a **two-dimensional free** field theory.
  - **Master formulas** exist for  **$n$ -point  $g$ -loop** amplitudes.
- **String theory** is **first-quantized**: the number of string loops is **fixed** at the outset.
  - Computations are performed in a  **$d = 2$**  field theory on a **Riemann surface** of fixed **genus  $g$** .
- **String theory** has an infinite number of **massive states**.
  - Masses are **multiples** of the **string tension**  $M_n^2 \propto n/\alpha' \propto nT$ .
  - **Tuning** the limit  $\alpha' \rightarrow 0$  for different strings one may get **different effective field theories**.

- In the **field theory limit**  $\alpha' \rightarrow 0$  Riemann surfaces **degenerate** into **Feynman-like graphs**
  - Only **massless** (or lowest-lying) **excitations circulate** in the loops.
  - **The  $g$ -loop string diagram generates all** field theory diagrams from different **corners of moduli space**.
  - Is it **practical**?



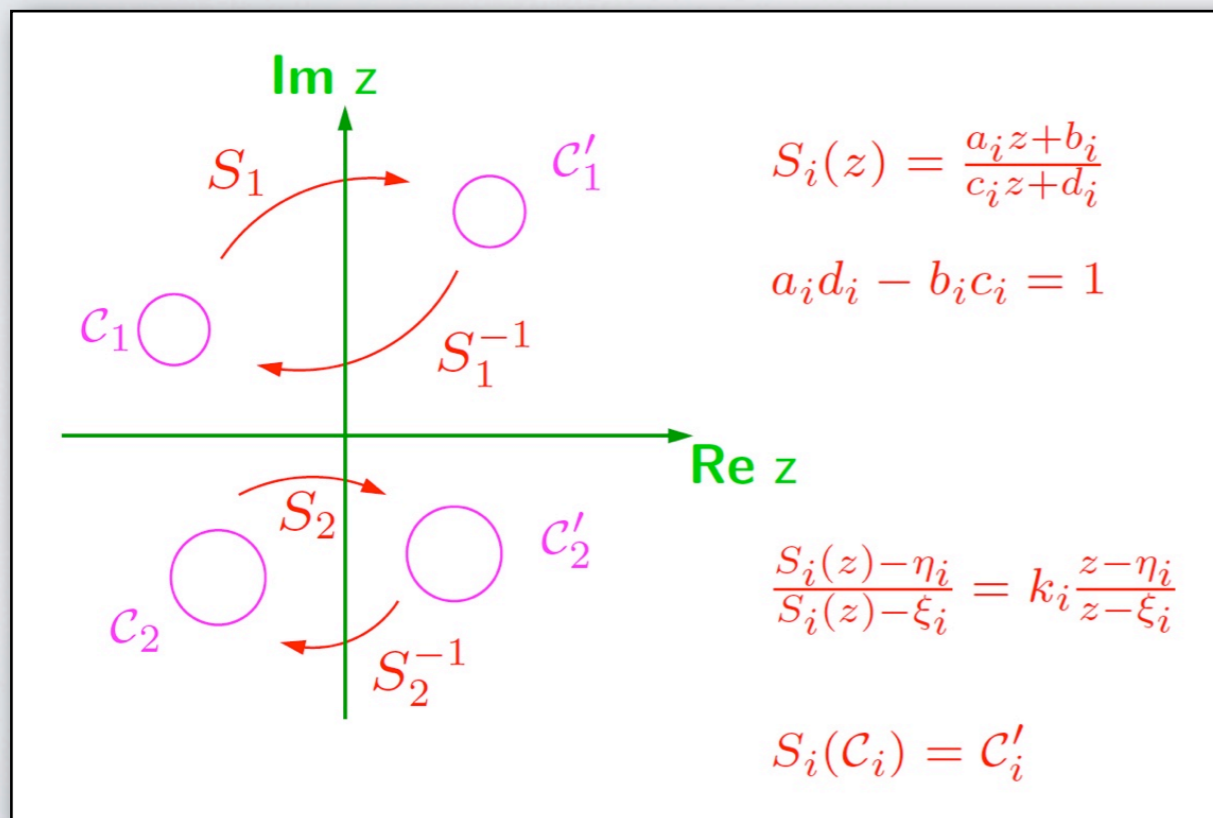
Different limit of a single string diagram

# The Schottky parametrization

- A **Riemann surface** of genus  $g$  can be represented by **cutting** and **identifying**  $g$  pairs of **circles** on the Riemann **sphere**, via **projective transformations**  $S_i$ .

  - The **Riemann surface** is then  $\Sigma_g = (\mathbf{C} \cup \infty) / \mathcal{S}_g$  where  $\mathcal{S}_g$  is the genus- $g$  **Schottky group** generated by the transformations  $S_i$ .
- Each projective transformation has **three complex parameters**, chosen as its **fixed points**  $\eta_i$  and  $\xi_i$ , and the **multiplier**  $k_i$ .

  - The **multipliers**  $k_i$  are proportional to the **radii** of the **circles**  $C_i$  and they drive the **field theory limit** as  $k_i \rightarrow 0$ .



- The **shape** of the genus- $g$  Riemann surface is determined by  **$3g - 3$  complex moduli** (**subtracting** one overall projective transformation on the sphere)

  - The number of moduli **matches** the number of **propagators** for  $g$ -loop **vacuum bubbles** with **cubic** vertices.
  - Geometric objects** on the Riemann surface can be expressed as **series** over the **Schottky group**.

# Geometric objects

The string **operator formalism** provides **explicit constructions** for geometric objects defined on Riemann surfaces, in terms of **series** on the **Schottky group**.

Let  $T_\alpha = S_i^a \cdot S_j^b \cdot \dots$  be an **element** of the Schottky group. Then one defines

**Abelian Differentials:** 
$$\omega_\mu = \sum_\alpha^{(\mu)} \left( \frac{1}{z - T_\alpha(\eta_\mu)} - \frac{1}{z - T_\alpha(\xi_\mu)} \right) dz$$

**Period Matrix:** 
$$\tau_{\mu\nu} = \frac{1}{2\pi i} \int_{b_\nu} \omega_\mu(z)$$

**Prime Form:** 
$$E_g(z, w) \sqrt{dzdw} = (z - w) \hat{\prod}_\alpha \frac{z - T_\alpha(w)}{z - T_\alpha(z)} \frac{w - T_\alpha(z)}{w - T_\alpha(w)}$$

**Scalar Propagator:** 
$$\mathcal{G}_g(z, w) = \log [E_g(z, w)] - \frac{1}{2} \int_z^w \omega_\mu \left[ (2\pi \mathbf{Im}\tau)^{-1} \right]^{\mu\nu} \int_z^w \omega_\nu$$

- In the **field theory limit** only a **handful** of Schottky group elements contribute.
- Relevant terms are **easily generated** with available software for symbolic manipulations.

# Master formulas

**String amplitudes** are computed by fixing the quantum numbers of the **external states** and then evaluating **correlation functions** of the corresponding **vertex operators** in the **two-dimensional** theory at the relevant **genus**.

Since the **2-d** theory is **free, closed form** results can be obtained for **g-loop, M-point** amplitudes. For example for **open string gluon states** one finds

$$A_{M,1}^{(g)}(\epsilon_1, p_1; \dots; \epsilon_N, p_N) = C_g \mathcal{N}^M \int [dm]_g^M \prod_{i < j} \exp \left[ 2\alpha' p_i \cdot p_j G_g(z_i, z_j) \right] \\ \times \left\{ \exp \left[ \sum_{i \neq j} \sqrt{2\alpha'} \epsilon_i \cdot p_j \partial_{z_i} G_g(z_i, z_j) + \frac{1}{2} \sum_{i \neq j} \epsilon_i \cdot \epsilon_j \partial_{z_i} \partial_{z_j} G_g(z_i, z_j) \right] \right\}_{\text{m.l.}}$$

Integration is over a **fundamental region** of the **g-loop moduli space**. **Normalization** can be computed in terms of the **string slope** and **coupling**, and the space-time dimension **d**.

$$C_g = (2\pi)^{-dg} (2\alpha')^{-d/2} g_s^{2g-2} \quad ; \quad \mathcal{N} = 2g_s (2\alpha')^{d/4-1/2}$$

The **conformal properties** of the **scalar propagator** can be **fixed** by a choice of **local coordinates**  $V_i(z)$  around the **punctures**: one writes

$$G_g(z_i, z_j) = \mathcal{G}_g(z_i, z_j) - \frac{1}{2} \log V_i'(0) - \frac{1}{2} \log V_j'(0)$$

where  $V_i(0)$  is required to have **conformal weight**  $w = -1$



# Pluses and minuses

## Remarkably ...

- Such formulas **exist**: no such results in field theory.
- Quantum numbers are **well-managed**:
  - \* **color decomposition** is already performed via **Chan-Paton** factors;
  - \* **loop momentum integration** is already **performed**, so that **helicity methods** are immediately **applicable**.
- Limited **off-shell continuation** is possible: the **gauge** chosen by string theory can be **identified**.
- While the **full** perturbative string amplitude is **not completely defined**, the field theory limit is **algorithmically implementable**.

## However ...

- Only a **limited set** of quantum field theories can be **reached**: massless gauge theories, gravity, unbroken SUSY.
  - \* Non-supersymmetric fermions are **difficult** to include.
  - \* Theories with **several mass scales** (SM ...) cannot be handled
- The problem is reduced to the computation of '**scalar integrals with numerators**': the method is **not competitive** with generalized unitarity in terms of speed.
- It is however **still interesting** within string theory and for its own sake:
  - \* **non-perturbative** applications via D-branes;
  - \* **new structures** at high loop order;
  - \* **dualities** ...

# ONE-LOOP GLUON AMPLITUDES

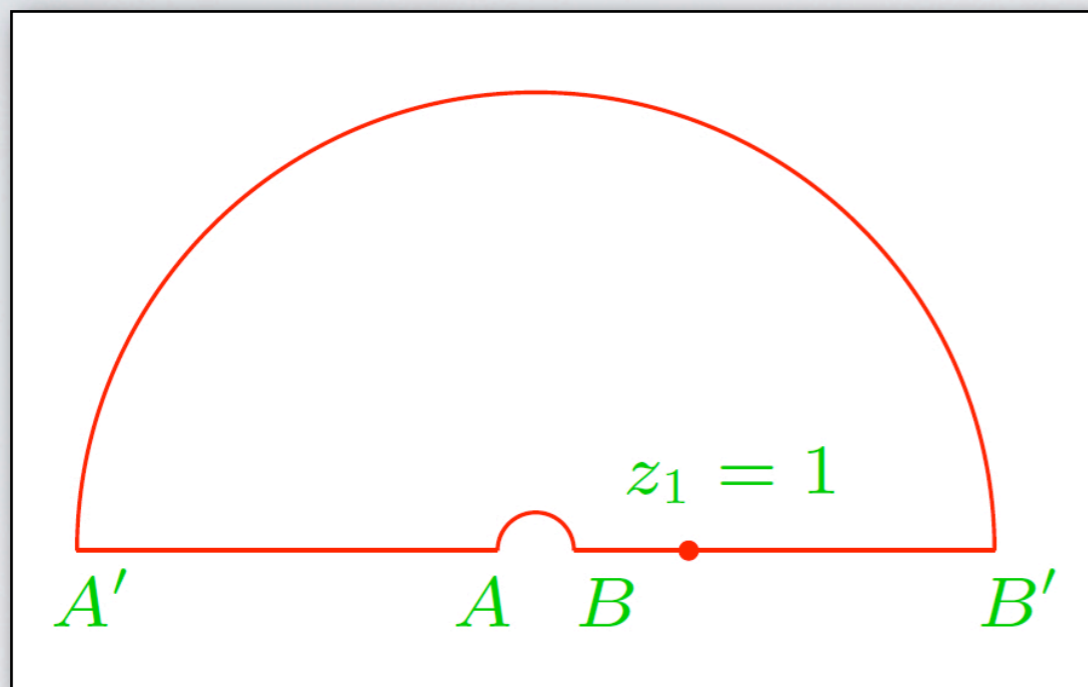


# The annulus

At **one loop** the Schottky group has only **one generator**: series and products over the group become **ordinary** Taylor series and products over **integers**.

One may **standardize** the representation of the annulus by **fixing the projective gauge** as

$$\eta = 0 \quad ; \quad \xi = \infty \quad ; \quad z_1 = 1$$



The cut annulus in the Schottky representation

The **Schottky generator** is then simply

$$S(z) = kz$$

which implies

$$B = -A = \sqrt{k}$$

$$B' = -A' = -1/\sqrt{k}$$

**External** states are **cyclically ordered** along either one of the boundaries, **AA'** or **BB'**.

The **integration region** is determined by symmetry and **modular invariance**:

- The transformation  $k \rightarrow 1/k$  does not affect the geometry
- One may map  $1 < z_i < 1/\sqrt{k} \rightarrow k < z_i < \sqrt{k}$

Thus one may **simply use**

$$0 < k < z_i < 1$$

# One-loop master formula

The **ingredients** of the one-loop **master formula** for **gluons** are easily determined

**Measure** of integration

$$[dm]_1^M = \frac{1}{k^2} \prod_{n=1}^{\infty} (1 - k^n)^{2-d} \left( -\frac{\log k}{2} \right)^{-\frac{d}{2}} \prod_{i=2}^M dz_i \Theta(z_i - z_{i+1})$$

**Matching** the string and the strong coupling, from tree level

$$g_s = \frac{1}{2} g_d (2\alpha')^{1-d/4}$$

The scalar **propagator**

$$G(z_i, z_j) = \log \left( \left| \sqrt{\frac{z_i}{z_j}} - \sqrt{\frac{z_j}{z_i}} \right| \right) + \frac{1}{2 \log k} \left( \log \frac{z_i}{z_j} \right)^2 + \log \left[ \prod_{n=1}^{\infty} \frac{\left( 1 - k^n \frac{z_j}{z_i} \right) \left( 1 - k^n \frac{z_i}{z_j} \right)}{(1 - k^n)^2} \right]$$

where the choice

$$V_i'(0) = (\omega(z_i))^{-1} = z_i$$

insures **modular invariance**

$$G(z_i/z_j; k) = G(z_j/z_i; k) = G(kz_i/z_j; k)$$

# The field theory limit

- From the string **operator formalism** we know that **Laurent expansion** of the integrand in powers of  $k$  counts the **mass level** of the state propagating in the loop.

$$k^{-2} \longrightarrow \text{tachyon} ; \quad k^{-1} \longrightarrow \text{gluon} ; \quad \dots$$

- The master formula has an **overall power** of  $\alpha'$ . **String moduli** defining the shape of the surface **must be expressed in units** of  $\alpha'$ , in order to take the limit  $\alpha' \rightarrow 0$

- Hint: **measure** of integration is  $d \log k$  ...

- Pedestrian** field theory limit (**exact** for scalars):

$$\log k = -\frac{t}{\alpha'} \quad ; \quad \log z_i = -\frac{t_i}{\alpha'}$$

- Note:  $t$  and  $t_i$  will be identified with **sums of Schwinger parameters** associated with **propagators** around the loop.

- For **gluons**, the overall **power**  $p$  of  $\alpha'$  after the change of variables is **not uniform**: instead,  $-M/2 < p < 0$ . One **must locate** all **further sources** of positive powers of  $\alpha'$ .

- Four-point** vertices  $(t_i - t_{i-1})/\alpha' = \mathcal{O}(1)$

- Expansion** of the exponential  $\exp \left[ 2\alpha' p_i \cdot p_j G_{ij} \right] \longrightarrow \exp \left[ c_0(t_i) + \alpha' c_1(t_i) \right]$

# Some results

- **One-loop diagrammatics** is fully **developed**.
  - À la **Bern-Kosower** (**no quartic vertex** topology).
  - **Direct** field theory limit **distinguishes** cubic and quartic vertices, **irreducible** and **reducible** topologies.

- **Off-shell continuation**, with identification of **individual topologies**, establishes the **gauge choice** naturally performed by string theory
  - For **irreducible** topologies: **Background Field Feynman** gauge
  - For **reducible** topologies **tree** subdiagrams are computed in the **Gervais-Neveu non-linear** gauge

$$S_{GN} = \int d^d x \left\{ -\frac{1}{4} \text{Tr} (F^2) - \frac{1}{2} \text{Tr} \left[ (\partial \cdot A - ig_d A^2)^2 \right] \right\}$$

- **Bosonic string theory** is well-defined only in the **critical** dimension  $d = 26$ . This has **no consequence** in the field theory limit: amplitudes have the **correct**  $d$  dependence (**dimensional regularization** à la 't Hooft-Veltman).

- **Bosonic string theory** has a **tachyon**. It can be **decoupled by hand**. Tachyons in loops have **IR divergences** not regulated dimensionally. Tachyon effects remain as **contact interactions**. Tachyon amplitudes **can be used** to **compute scalar amplitudes** in field theory by the **replacement**

$$dx/x^2 \longrightarrow \left[ \exp(\alpha' m^2 \log x) \right] dx/x$$

# TWO-LOOP AMPLITUDES

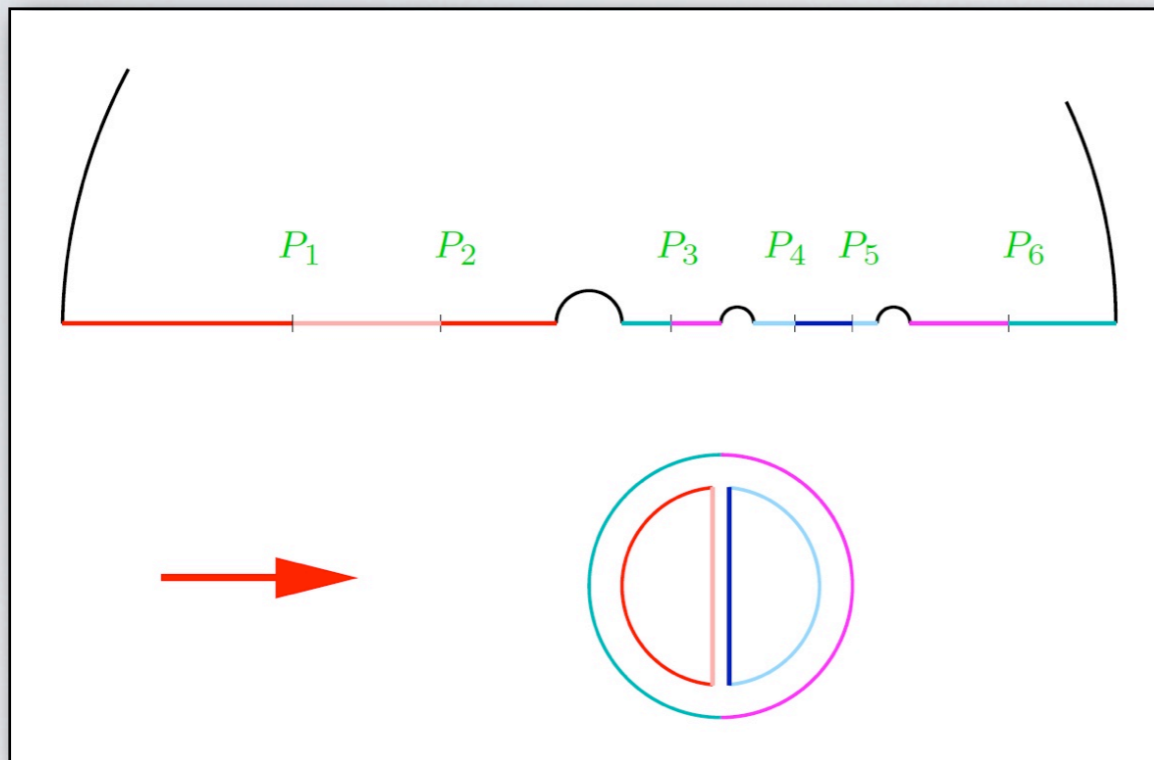


# The double annulus

At **two loops** the Schottky group has **two generators**: however **expanding** in powers of the **multipliers** remains **simple** since  $S_i ( S_i (z) )$  contributes to order  $k_i^2$ .

One may **standardize** the representation of the double annulus by **choosing the gauge** as

$$\eta_1 = 0 \quad ; \quad \xi_1 = \infty \quad ; \quad \xi_2 = 1$$



The cut double annulus in the Schottky representation

The **Schottky generators** are then

$$S_1(z) = k_1 z$$

and

$$S_2(z) = \frac{\eta(1-z) + k_2(z-\eta)}{(1-z) + k_2(z-\eta)}$$

It is possible to **identify** precisely on **which propagator** and on **which boundary** the punctures are **inserted**

Insertion on **different boundaries** yields **different expressions** for the **integrand** of the amplitude, but the results are **related by modular transformations**, providing highly **nontrivial checks** on the field theory limit.



# Scalar amplitudes

Some **ingredients** of the two-loop **master formula** for **scalars** are given by

- 📌 **Matching** the string and the scalar coupling, from tree level

$$g_s = \frac{1}{4} \lambda (2\alpha')^{(6-d)/4}$$

- 📌 The scalar **propagator** to **leading order** in the multipliers  $k_1$  and  $k_2$

$$\mathcal{G}_2(z_i, z_j) = \log(|z_i - z_j|) + \frac{1}{2} \frac{\log k_1 \log k_2 - \log^2 S}{\log^2 T \log k_2 + \log^2 U \log k_1 - 2 \log T \log U \log S}$$

where

$$S = \eta_2 \quad ; \quad T = \frac{z_i}{z_j} \quad ; \quad U = \frac{(z_j - \eta_2)(z_i - 1)}{(z_i - \eta_2)(z_j - 1)}$$

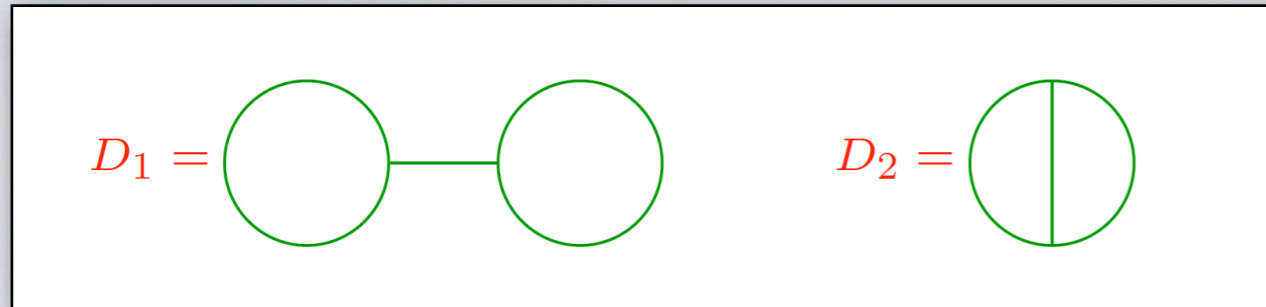
- 📌 The choice of **local coordinates** around the punctures

$$V'_i(0) = \left[ a_1 \omega_1(z_i) + a_2 \omega_2(z_i) \right]^{-1}$$

with  $a_i$  **chosen** according to the **boundary** of insertion, insures **modular invariance**, for example under the transformation **exchanging** the two **inner** boundaries

$$z \rightarrow \frac{z - \eta_2}{z - 1}$$

# Vacuum bubbles



Topologies for scalar two-loop vacuum bubbles

**Leading regions** in the field theory limit arise as  $k_i \rightarrow 0$ , and

$$\eta_1 \rightarrow 0 \quad \text{or} \quad \eta_1 \rightarrow 1$$

The fixed point  $\eta_1$  plays the role of '**distance between the loops**'.

In the **two** relevant **regions** dimensionful **proper-time variables** are defined as

$$\begin{aligned} \eta_1 \rightarrow 1 : & \quad k_i = e^{-t_i/\alpha'} \quad ; \quad 1 - \eta_1 = e^{-t_3/\alpha'} \\ \eta_1 \rightarrow 0 : & \quad q_i \equiv k_i/\eta_1 = e^{-t_i/\alpha'} \quad ; \quad q_3 \equiv \eta_1 = e^{-t_3/\alpha'} \end{aligned}$$

The **integration region** is found requiring that Schottky circles **do not overlap**, and simplifies in the field theory limit. **Regulating tachyon** double poles by treating  $m^2$  as **generic** one finds

$$\begin{aligned} D_1 &= \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^\infty dt_2 \int_0^{t_2} dt_1 e^{-m^2(t_1+t_2+t_3)} (t_1 t_2)^{-d/2} \\ D_2 &= \frac{N^3}{(4\pi)^d} \frac{g^2}{32} \int_0^\infty dt_3 \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 e^{-m^2(t_1+t_2+t_3)} (t_1 t_2 + t_1 t_3 + t_2 t_3)^{-d/2} \end{aligned}$$

which as expected **agrees with field theory**, including **color** and **symmetry** factors.

# EFFECTIVE ACTIONS



# Field theory: scalars

**Effective actions** are useful to study the **geometry of moduli space** through 'vacuum bubbles'.

Consider coupling an **adjoint scalar** to a constant **background field**.

$$\mathcal{L} = \text{Tr} \left[ D_\mu \Phi D^\mu \Phi - m^2 \Phi^2 + \frac{2}{3} \lambda \Phi^3 \right] .$$

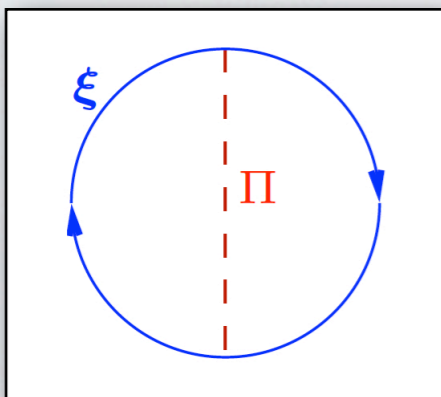
Using a **pseudo-abelian** constant gauge field (chromo-magnetic) field one may write

$$\mathcal{A}_{ab}^\mu = A^\mu \delta_{a,N} \delta_{b,N} \quad ; \quad A_\mu = B x_1 g_{\mu 2} \quad \Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \Pi(x) & \xi(\mathbf{x}) \\ \xi^\dagger(\mathbf{x}) & \sigma(x) \end{pmatrix}$$

In this case the **scalar charged propagator** can be **computed exactly**

$$G_\xi(x, y) = \frac{e^{-iB(x_1+y_1)(x_2-y_2)/2}}{(4\pi)^{d/2}} \int_0^\infty dt e^{-m^2 t} t^{-d/2+1} \frac{B}{\sinh(Bt)} \\ \times \exp \left[ \frac{(x_0 - y_0)^2 - (\mathbf{x}_\perp - \mathbf{y}_\perp)^2}{4t} - \frac{B(x_1 - y_1)^2 - (x_2 - y_2)^2}{4 \tanh(Bt)} \right] .$$

Formal (unrenormalized) expressions for **two-loop vacuum bubbles** are readily computed



$$W_{\xi\Pi}^{(2)}(m, B) = -i V_d \frac{\lambda^2}{(4\pi)^d} \frac{(N-1)^2}{4} \int_0^\infty dt_1 dt_2 dt_3 e^{-m^2(t_1+t_2+t_3)} \Delta_0^{-\frac{d}{2}+1} \Delta_B^{-1}$$

$$\Delta_0 = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad \Delta_B = \frac{1}{B^2} \sinh(Bt_2) \sinh(Bt_3) + \frac{t_1}{B} \sinh[B(t_2 + t_3)] .$$

A two-loop charged diagram

# Field theory: gauge bosons

A more **challenging** and **interesting** calculation is **pure Yang-Mills** theory,

$$\mathcal{L} = -\frac{1}{2} \text{Tr} \left[ F_{(A+Q)}^2 \right] - \frac{1}{\xi} \text{Tr} \left[ G^2(A, Q) \right] + \mathcal{L}_{\text{ghost}}$$

With string theory in mind, we pick an **intricate gauge** (**Bern, Dunbar**), the **background field** version of the **Gervais-Neveu** gauge

$$G(A, Q) = D_{\mu}^{(A)} Q^{\mu} + \frac{i}{2} \alpha g \{Q_{\mu}, Q^{\mu}\}$$

In this form, it is (almost) the **most general** gauge choice **compatible** with the BF method.

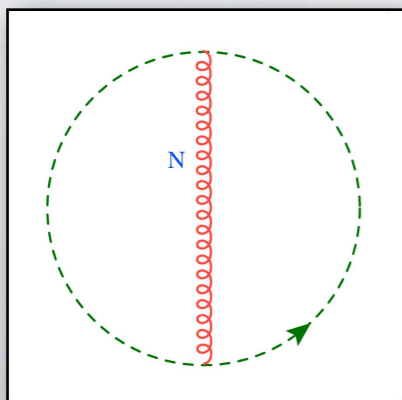
As before, we take a **block-diagonal** gauge field, non-trivial in a **fixed plane**.

$$A_{\mu}^{(i)} = \delta_{a_i b_i} B^{(i)} x_1 g_{\mu 2} \quad \longrightarrow \quad F_{12}^{(A)} = \text{diag} \left\{ B^{(i)} \right\}$$

The gluon propagator has **charged polarizations**, which can be **diagonalized**

$$G_{ij}^{+}(x, y) = -g^{+-} G_{\xi}(x, y), \quad \text{with } B \rightarrow B_i - B_j, \quad e^{-m^2 t} \rightarrow e^{-2(B_i - B_j)t}$$

Notwithstanding a (well-known) **instability**, computations can be **formally carried out**:



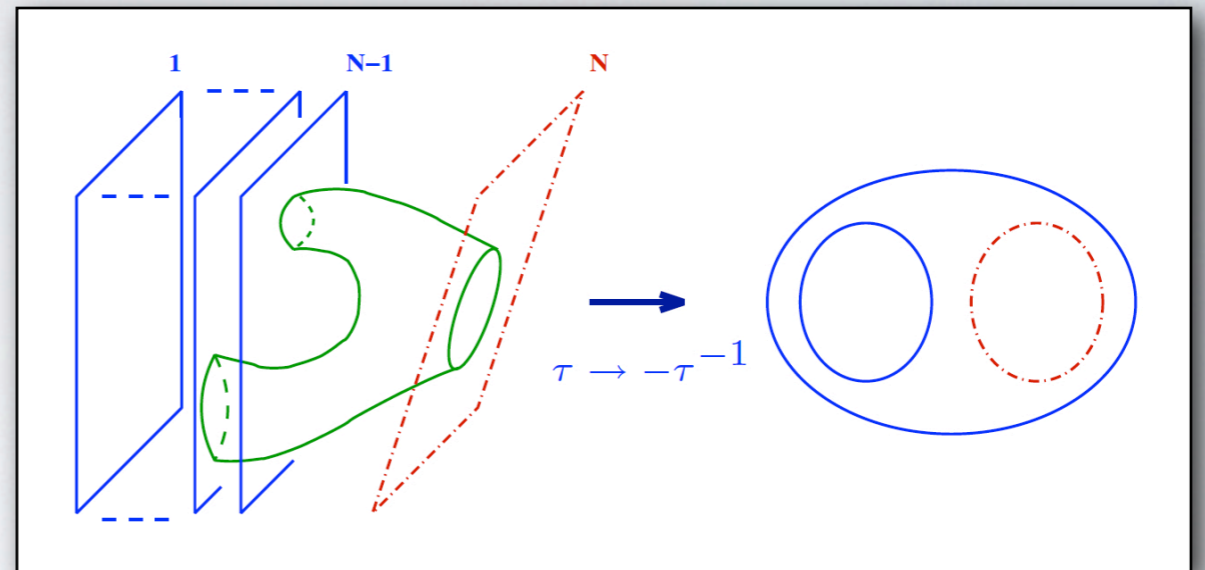
$$= \frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d \frac{1 + \alpha^2}{2} (d - 2) \int_0^{\infty} dt_1 dt_2 dt_3 \frac{t_1 + t_2 + t_3}{\Delta_0^{d/2} \Delta_B \prod_{i=1}^3 \cosh(B_i t_i)}$$

# String theory: master formulas

The technology to compute **multi-loop** string **effective actions** in these backgrounds was recently **developed** (Russo, Sciuto).

**Basic fact:** constant backgrounds simply modify the **boundary conditions** for the **world-sheet fields**

$$\left[ \partial_\sigma X^i + i \partial_\tau X^j F_j^{i(A)} \right]_{\sigma=0} = 0$$



Dual pictures for the two-loop string effective action

**Twisted** boundary conditions lead to **new geometric objects** on the world sheet: **twisted determinants**, Prym **differentials**, a twisted **period matrix**. They are all **computable** in the Schottky parametrization. The **g-loop partition function** reads

$$Z_\epsilon(g) = \frac{e^{2\pi i \epsilon_g} - 1}{\prod_{\mu=1}^g \cos \pi \epsilon_\mu} C_g \int [dm]_g^0 e^{-i\pi \epsilon \cdot \tau \cdot \epsilon} \frac{\det(\tau)}{\det(\tau_\epsilon)} \mathcal{R}_g(k_\alpha, \epsilon \cdot \tau)$$

where a vector of **dimensionless fields** was defined by  $\tan(\pi \epsilon_i) = 2\pi \alpha' (B_i - B_{i+1})$

**New geometric objects** contributing to the partition function, such as

$$\mathcal{R}_g(k_\alpha, \epsilon) = \frac{\prod_{\alpha'} \prod_{n=1}^{\infty} (1 - k_\alpha^n)^2}{\prod_{\alpha'} \prod_{n=1}^{\infty} (1 - e^{-2\pi i \epsilon \cdot N_\alpha} k_\alpha^n) (1 - e^{2\pi i \epsilon \cdot N_\alpha} k_\alpha^n)}$$

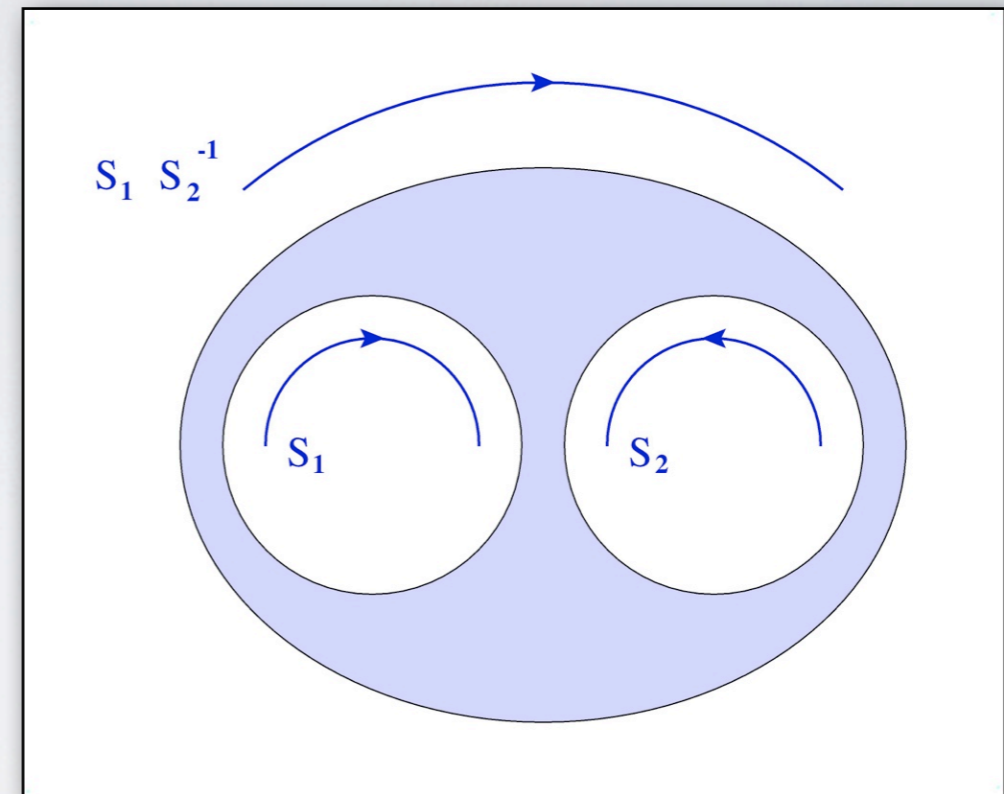
can still be **computed perturbatively** in the field theory limit.

# Picking the right variables

The **difficulty** in extending two-loop calculations **beyond** the string **ground state** can be traced to a **failure** of the **pedestrian choice** of **variables** for the field theory limit

A **better choice** is dictated by geometry, and **modular invariance**: each **boundary** must be **decomposed** as the **product** of **two** propagators in a modularly **covariant** way

One can **postulate** an **exact factorization** of the **multipliers** associated with each boundary as



Schottky actions on the two-loop annulus

$$k(S_1) \equiv p_1 p_3, \quad k(S_2) \equiv p_2 p_3, \quad k(S_1 \cdot S_2^{-1}) \equiv p_1 p_2,$$

The third definition appears to lead to complicated **square-root** singularities. **Remarkably**, it can be **simply inverted** as

$$k_1 = p_1 p_3, \quad k_2 = p_2 p_3, \quad \eta = \frac{(1 + p_1)(1 + p_2)p_3}{(1 + p_3)(1 + p_1 p_2 p_3)}$$

The **field theory limit** for this topology is then driven by **expanding** in  $p_i$ , using

$$\log p_i \equiv -t_i/\alpha'$$

# Picking diagrams

Further progress stems from reconstructing the origin of individual contributions to the string partition function.

- Contributions to the measure arising from the determinant of the  $(b-c)$  system (world-sheet ghosts) correspond to the propagation of space-time ghosts (Polchinski).
- Contributions of world-sheet scalars correspond to un-magnetized gluon propagation
- One may distinguish the space-time dimension from the string critical dimension, allowing for propagation of adjoint scalars left over from extra-dimensional gluons.



# Picking diagrams

Further progress stems from **reconstructing** the **origin** of individual contributions to the the string partition function.

- **Contributions** to the measure arising from the the determinant of the **(b-c) system** (**world-sheet ghosts**) correspond to the propagation of **space-time ghosts** (Polchinski).
- **Contributions** of **world-sheet scalars** correspond to **un-magnetized gluon** propagation
- One may distinguish the **space-time dimension** from the string **critical dimension**, allowing for propagation of **adjoint scalars** left over from extra-dimensional gluons.

Consider the **untwisted** contribution to the **two-loop integrand** for the effective action

$$\mathcal{I}_2 (\epsilon \rightarrow 0) = \frac{dk_1 dk_2 d\eta}{k_1^2 k_2^2} \frac{(1 - k_1)^2 (1 - k_2)^2}{(1 - \eta)^2} \left[ \det(\tau) \right]^{-d/2+1} \prod_{\alpha}^{\prime} \left[ \prod_{n=1}^{\infty} (1 - k_{\alpha}^n)^{-d} \prod_{n=2}^{\infty} (1 - k_{\alpha}^n)^2 \right]$$

# Picking diagrams

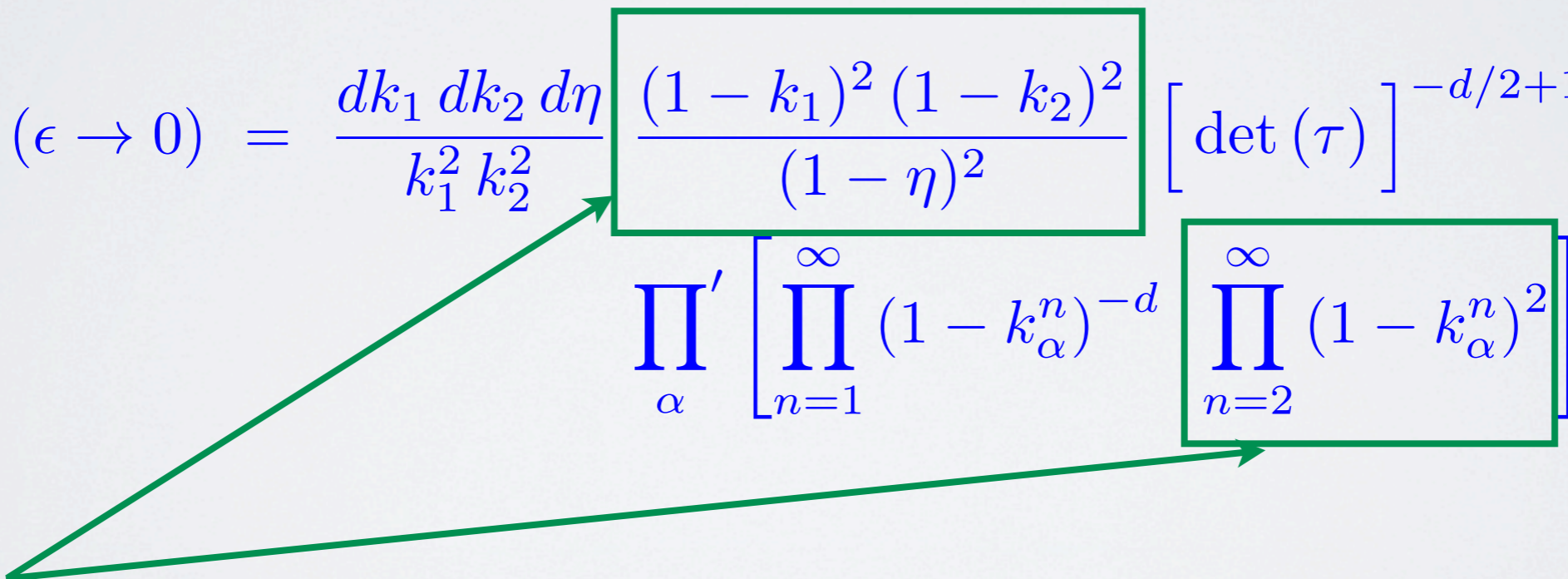
Further progress stems from **reconstructing** the **origin** of individual contributions to the the string partition function.

- **Contributions** to the measure arising from the the determinant of the **(b-c) system** (**world-sheet ghosts**) correspond to the propagation of **space-time ghosts** (Polchinski).
- **Contributions** of **world-sheet scalars** correspond to **un-magnetized gluon** propagation
- One may distinguish the **space-time dimension** from the string **critical dimension**, allowing for propagation of **adjoint scalars** left over from extra-dimensional gluons.

Consider the **untwisted** contribution to the **two-loop integrand** for the effective action

$$\mathcal{I}_2 (\epsilon \rightarrow 0) = \frac{dk_1 dk_2 d\eta}{k_1^2 k_2^2} \frac{(1 - k_1)^2 (1 - k_2)^2}{(1 - \eta)^2} \left[ \det(\tau) \right]^{-d/2+1} \prod_{\alpha} \left[ \prod_{n=1}^{\infty} (1 - k_{\alpha}^n)^{-d} \prod_{n=2}^{\infty} (1 - k_{\alpha}^n)^2 \right]$$

**Ghosts**



# Picking diagrams

Further progress stems from reconstructing the origin of individual contributions to the string partition function.

- Contributions to the measure arising from the determinant of the  $(b-c)$  system (world-sheet ghosts) correspond to the propagation of space-time ghosts (Polchinski).
- Contributions of world-sheet scalars correspond to un-magnetized gluon propagation
- One may distinguish the space-time dimension from the string critical dimension, allowing for propagation of adjoint scalars left over from extra-dimensional gluons.

Consider the untwisted contribution to the two-loop integrand for the effective action

$$\mathcal{I}_2 (\epsilon \rightarrow 0) = \frac{dk_1 dk_2 d\eta}{k_1^2 k_2^2} \frac{(1 - k_1)^2 (1 - k_2)^2}{(1 - \eta)^2} [\det(\tau)]^{-d/2+1} \prod_{\alpha} \left[ \prod_{n=1}^{\infty} (1 - k_{\alpha}^n)^{-d} \prod_{n=2}^{\infty} (1 - k_{\alpha}^n)^2 \right]$$

**Ghosts**

**Un-magnetized gluons**

# Picking diagrams

Further progress stems from reconstructing the origin of individual contributions to the string partition function.

- Contributions to the measure arising from the determinant of the *(b-c) system* (*world-sheet ghosts*) correspond to the propagation of *space-time ghosts* (Polchinski).
- Contributions of *world-sheet scalars* correspond to *un-magnetized gluon* propagation
- One may distinguish the *space-time dimension* from the string *critical dimension*, allowing for propagation of *adjoint scalars* left over from extra-dimensional gluons.

Consider the *untwisted* contribution to the *two-loop integrand* for the effective action

$$\mathcal{I}_2 (\epsilon \rightarrow 0) = \frac{dk_1 dk_2 d\eta}{k_1^2 k_2^2} \frac{(1 - k_1)^2 (1 - k_2)^2}{(1 - \eta)^2} [\det(\tau)]^{-d/2+1} \prod_{\alpha} \left[ \prod_{n=1}^{\infty} (1 - k_{\alpha}^n)^{-d} \prod_{n=2}^{\infty} (1 - k_{\alpha}^n)^2 \right]$$

**Ghosts**

**Un-magnetized gluons**

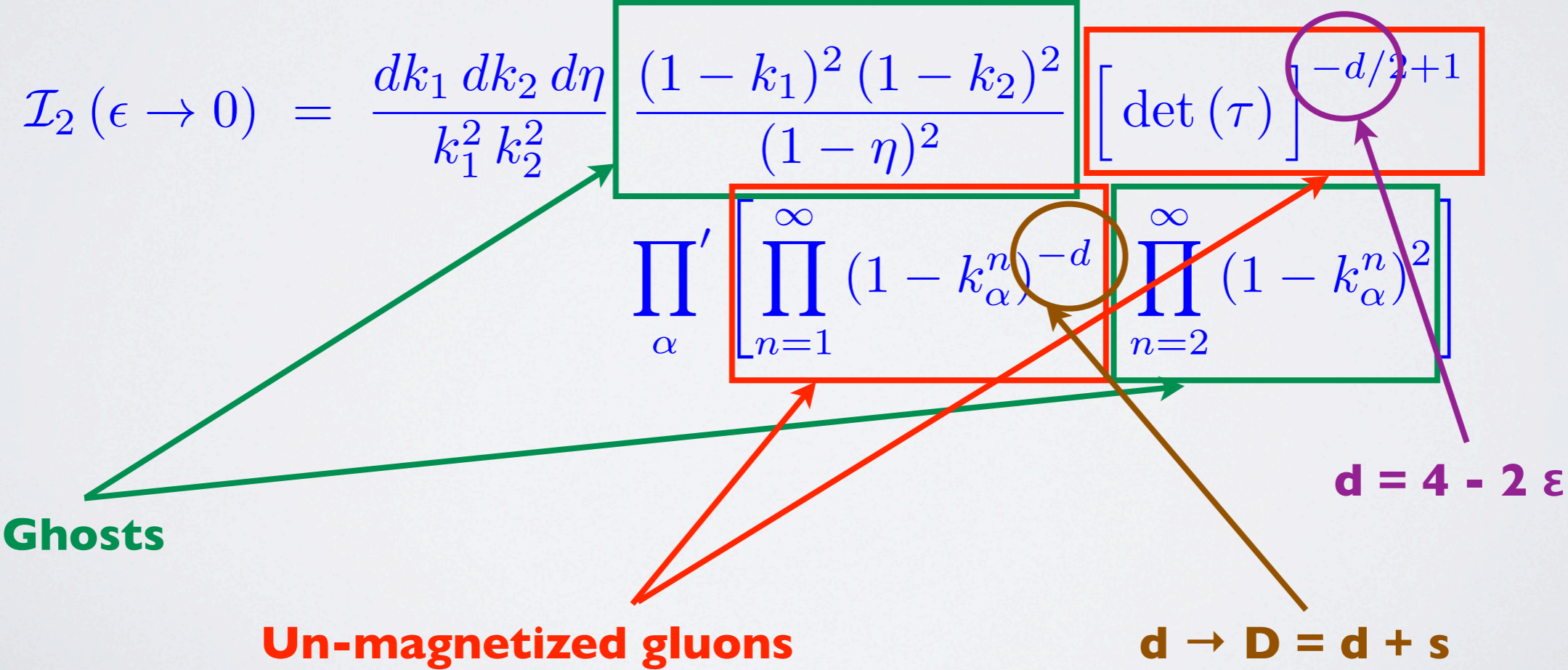
$d = 4 - 2 \epsilon$

# Picking diagrams

Further progress stems from reconstructing the origin of individual contributions to the the string partition function.

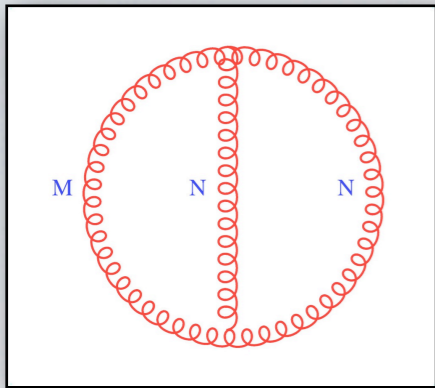
- Contributions to the measure arising from the the determinant of the  $(b-c)$  system (world-sheet ghosts) correspond to the propagation of space-time ghosts (Polchinski).
- Contributions of world-sheet scalars correspond to un-magnetized gluon propagation
- One may distinguish the space-time dimension from the string critical dimension, allowing for propagation of adjoint scalars left over from extra-dimensional gluons.

Consider the untwisted contribution to the two-loop integrand for the effective action



# Sample results

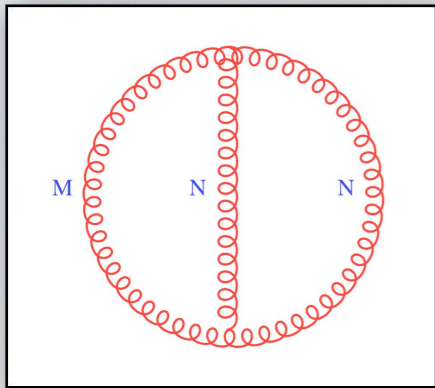
With this technology **all relevant two-loop diagrams** can be **extracted**. For example



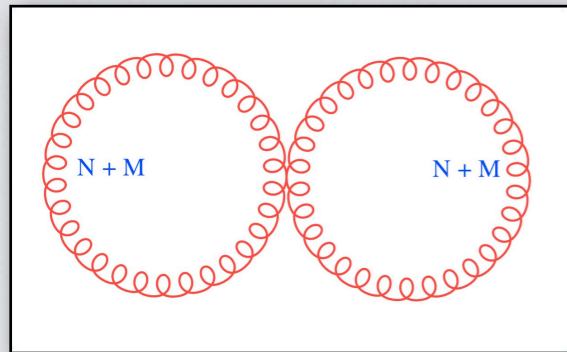
$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \Delta_B^2 \prod_{i=1}^3 \cosh^2 [B_i t_i]} \\ \times \left[ 2 \frac{\sinh(B_1 t_1)}{B_1} \cosh(2B_1 t_1 - B_2 t_2 - B_3 t_3) + \mathcal{O}(1 - \alpha^2) \right]$$

# Sample results

With this technology **all relevant two-loop diagrams** can be **extracted**. For example



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \Delta_B^2 \prod_{i=1}^3 \cosh^2 [B_i t_i]} \\ \times \left[ 2 \frac{\sinh(B_1 t_1)}{B_1} \cosh(2B_1 t_1 - B_2 t_2 - B_3 t_3) + \mathcal{O}(1 - \alpha^2) \right]$$

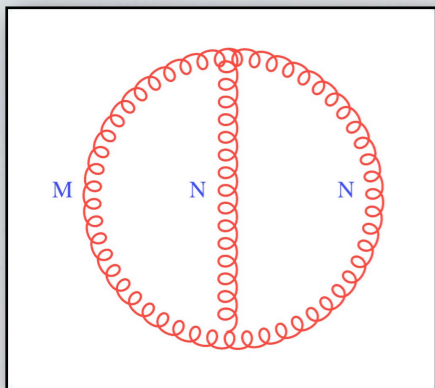


$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d \frac{1 + \alpha^2}{2} \int_0^\infty dt_1 dt_2 \frac{B_1 t_1 B_2 t_2}{\sinh(B_1 t_1) \sinh(B_2 t_2)} \\ \times \left[ (d-2) + 2 \cosh(2B_1 t_1 + 2B_2 t_2) + \mathcal{O}(1 - \alpha^2) \right]$$

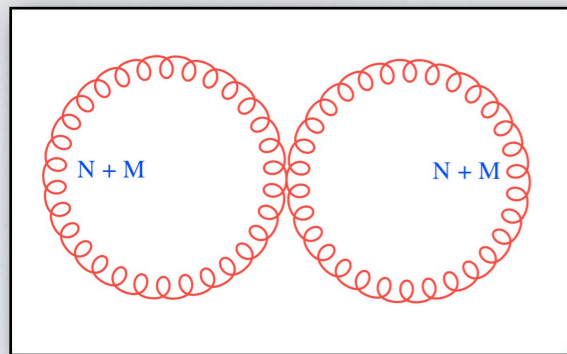
- Isolating the **four-point vertex** requires **integrating exactly** over  $\eta$  after expanding in  $k_i$ , **discarding** tachyon-related **singularities**.

# Sample results

With this technology **all relevant two-loop diagrams** can be **extracted**. For example



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \Delta_B^2 \prod_{i=1}^3 \cosh^2 [B_i t_i]} \times \left[ 2 \frac{\sinh(B_1 t_1)}{B_1} \cosh(2B_1 t_1 - B_2 t_2 - B_3 t_3) + \mathcal{O}(1 - \alpha^2) \right]$$



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 \frac{1 + \alpha^2}{\Delta_0^{d/2-1} \Delta_B^2} \int_0^\infty dt_1 dt_2 \frac{B_1 t_1 B_2 t_2}{\sinh(B_2 t_2)} \times \left[ (d-2) \frac{\sinh(B_1 t_1)}{B_1} \cosh(B_1 t_1 - B_2 t_2) + \mathcal{O}(1 - \alpha^2) \right]$$

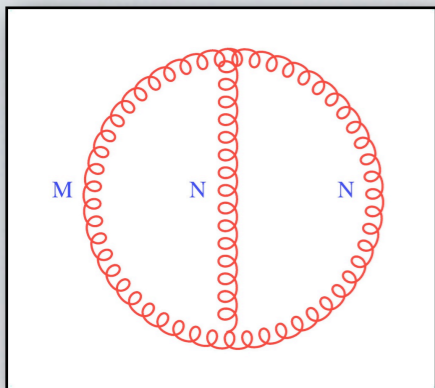
Preliminary ...

- Isolating the **four-point vertex** requires **integrating exactly** over  $\eta$  after expanding in  $k_i$ , **discarding** tachyon-related **singularities**. **Tests** for ambiguities **are underway**.

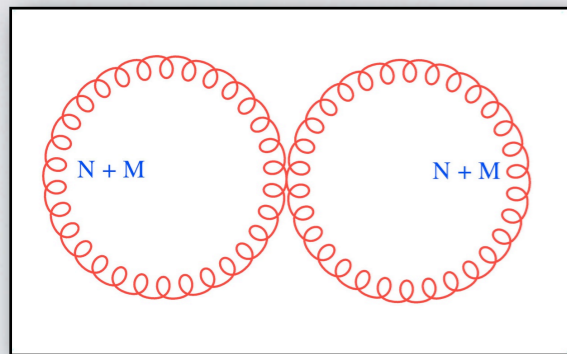


# Sample results

With this technology **all relevant two-loop diagrams** can be **extracted**. For example



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \Delta_B^2 \prod_{i=1}^3 \cosh^2 [B_i t_i]} \times \left[ 2 \frac{\sinh(B_1 t_1)}{B_1} \cosh(2B_1 t_1 - B_2 t_2 - B_3 t_3) + \mathcal{O}(1 - \alpha^2) \right]$$



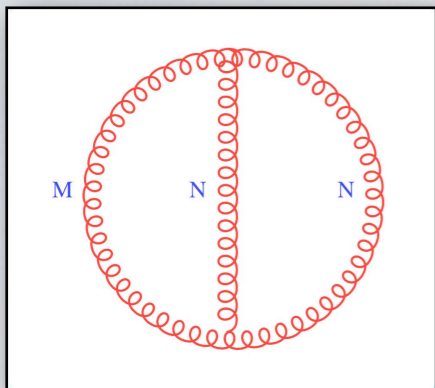
$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 \frac{1 + \alpha^2}{\int_0^\infty dt_1 dt_2} \frac{B_1 t_1 B_2 t_2}{\sinh(B_2 t_2)} \times \left[ (d-2) \frac{1}{\Delta_0^{d/2-1} \Delta_B^2} \cosh(B_1 t_1 - B_2 t_2) + \mathcal{O}(1 - \alpha^2) \right]$$

Preliminary ...

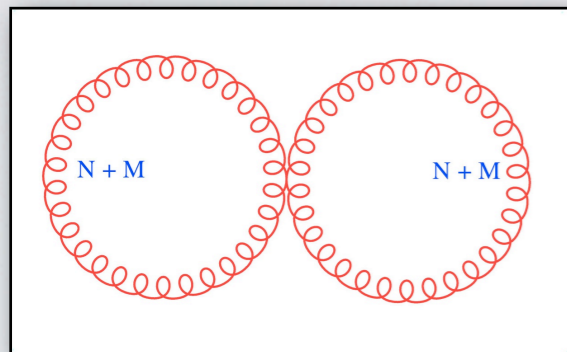
- Isolating the **four-point vertex** requires **integrating exactly** over  $\eta$  after expanding in  $k_i$ , **discarding** tachyon-related **singularities**. **Tests** for ambiguities **are underway**.
- **All** Schwinger parameter **integrand**s are **reproduced** by string theory with  $\alpha = 1$

# Sample results

With this technology **all relevant two-loop diagrams** can be **extracted**. For example



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 V_d (d-2) \int_0^\infty dt_1 dt_2 dt_3 \frac{1}{\Delta_0^{d/2-1} \Delta_B^2 \prod_{i=1}^3 \cosh^2 [B_i t_i]} \times \left[ 2 \frac{\sinh(B_1 t_1)}{B_1} \cosh(2B_1 t_1 - B_2 t_2 - B_3 t_3) + \mathcal{O}(1 - \alpha^2) \right]$$



$$= -\frac{g^2}{(4\pi)^d} N_1 N_2 N_3 \frac{1 + \alpha^2}{\int_0^\infty dt_1 dt_2} \frac{B_1 t_1 B_2 t_2}{\sinh(B_2 t_2)} \times \left[ (d-2) \frac{1}{\Delta_0^{d/2-1} \Delta_B^2} \mathcal{O}(1 - \alpha^2) \right]$$

Preliminary ...

- Isolating the **four-point vertex** requires **integrating exactly** over  $\eta$  after expanding in  $k_i$ , **discarding** tachyon-related **singularities**. **Tests** for ambiguities **are underway**.
- **All** Schwinger parameter **integrand**s are **reproduced** by string theory with  $\alpha = 1$
- **Integrals** are **highly divergent** and **IR singular**. To **test** for a **physical result** we have reproduced the finite **renormalized** two-loop effective action of **scalar electrodynamics** computed by **Ritus** in the **70's**.

# OUTLOOK



# To summarize

- 📌 Our understanding of the **field theory limit** of **perturbative string amplitudes** grows at **widely separated** time intervals.
- 📌 Pushing **beyond one loop** and beyond the string **ground state** proved **difficult** so far.
- 📌 Studying **effective actions** in constant background gauge fields at **multiloops** is now **possible** and **useful** to understand the **fragmentation of moduli space** in the field theory limit.
- 📌 We have made **significant progress**.
  - The **proper variables** to identify each vacuum bubble topology have been **identified**, in a manner generalizable to **higher genera**.
  - **Reconstructing** the origin of each factor of the string partition function it is **possible** to **identify** not only individual topologies but **individual diagrams**.
  - **String theory** naturally computes diagrams in a **specific gauge**: the **Gervais-Neveu Background Field** (GNBF) gauge with parameter  $\alpha = 1$ . This applies to **all genera**.
- 📌 Massless and massive **scalar** fields, massless **gauge bosons** and **ghosts** can be handled using the open **bosonic string**. **Gravitons** are expected to follow the same pattern in the **closed string** channel. **Fermions** must await further technical developments for **superstring** amplitudes.
- 📌 **Applications** can be envisaged to perturbative **dualities**, non-perturbative field theory effects (**D-branes and instantons**), string **phenomenology** and further **theory developments**.

THANK YOU!