Even More Perturbative QCD ...

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Abstract

These lectures describe the treatment of mass divergences in perturbative QCD, concentrating on hadron production in electron-positron annihilation. I perform an explicit and detailed one-loop calculation, and use it to infer some results and techniques valid to all orders in perturbation theory. I introduce some of the tools necessary to prove factorization theorems, and show how they can be used also to resum certain classes of logarithmic contributions to all orders in perturbation theory.

Ubi maior ... :

G. Sterman: An introduction to quantum field theory.
P. Nason: http://castore.mib.infn.it/~nason/misc/QCD...
M.L. Mangano: http://home.cern.ch/~mlm/talks/cern98...
G. Sterman: hep-ph/9606312.

– Parma, 05/09/2002 –

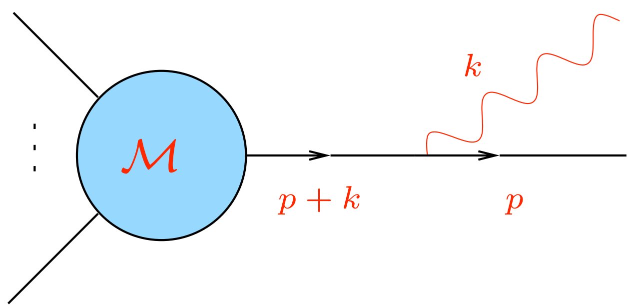
Outline

- On mass divergences in perturbative QCD
  - Mass divergences, low energies e long distances.
  - Cancellation for physical observables, KLN theorem.
  - Factorizable and IR/C safe observables in PQCD.
- An explicit example: $R_{e^+e^-}$
  - Definitions, cut diagrams, tree level.
  - $\mathcal{O}(\alpha_s)$: the calculation.
  - Approximations and observations.
- Other infrared finite quantities
  - Sterman–Weinberg jets.
  - Event shapes.
- Methods for all-order calculations
  - Singularities of Feynman diagrams and trapped surfaces.
  - Landau equations and Coleman-Norton picture.
  - Infrared power–counting and finiteness of $R_{e^+e^-}$.
- Factorization and resummation
  - Multi-scale problems and large logarithms.
  - From factorization to resummation.
  - Examples: the quark form factor; the thrust.

Mass divergences: qualitative discussion

- Fact: in quantum field theory, *two* kinds of divergences are associated with the presence of massless particles.
  - Infrared (IR). Emission of particles with vanishing four momentum ($\lambda_{DB} \to \infty$); in gauge theories only; they are present also when matter particles are massive.
  - Collinear (C). Emission of particles moving parallel to the emitter; they are present if all particles in the interaction vertex are massless.

- Example: a massless fermion emits a gauge boson in the final state.



$$\to \quad - \mathrm{i} g \overline{u}(p) \not\epsilon(k) t_a \frac{\mathrm{i}(\not p + \not k)}{(p+k)^2 + \mathrm{i}\varepsilon} \mathcal{M} \; ,$$

Singularities: $2p \cdot k = 2p_0 k_0 (1 - \cos\theta_{pk}) = 0$,

$\to k_0 = 0$ (IR); $\cos\theta_{pk} = 0$ (C).

Note: $p_0 = 0$ not a problem (singularities are always integrable).

- Origin of mass singularities
  - In covariant perturbation theory (p^μ conserved in every vertex; intermediate particles generally off–shell): the emitting fermion is on–shell, so that it can propagate indefinitely.
  - In time-ordered perturbation theory (all particles are on–shell, energy however is not generally conserved in the interaction vertices): the IR/C emission vertex conserves energy, so it can be placed at arbitrary distance from the primary process.
  - Mass divergences originate from physical processes happening at large distances.

- Available terapies .
  - The sickness is serious. Because of mass divergences, the S matrix cannot be constructed in the Fock space of quarks and gluons
  - Observe. Mass divergences are associated with the existence of experimentally indistinguishable, energy degenerate states. All physical detectors have finite resolution in energy and angle.
  - KLN Theorem. Physically measurable quantities (such as transition probabilities, cross sections, after summing coherently over all physically indistinguishable states) are finite, mass divergences cancel.

- KLN theorem.

  Consider a theory defined by its hamiltonian H, and let $\mathcal{D}_\epsilon(E_0)$ be the set of eigenstates of H characterized by energies $E_0 - \epsilon \leq E \leq E_0 + \epsilon$, with $\epsilon \neq 0$. Let $P(i \to j)$ be the transition probability per unit volume and per unit time from eigenstate i to eigenstate j. Then the quantity

  $$P(E_0, \epsilon) \equiv \sum_{i,j \in \mathcal{D}_\epsilon(E_0)} P(i \to j)$$

  is finite in the massless limit to all orders in perturbation theory

- Note. In an asimptotically free field theory the limit $m \to 0$ and the high-energy limit formally coincide. Masses acquire a scale dependences such that $m^2(\mu^2) \to 0$ as $\mu^2 \to \infty$.

- The situation in perturbative QCD
  - Long distance ($d \gtrsim 1\mathrm{fm}$), or low energy ($E \lesssim 1\mathrm{GeV}$) physics is not perturbatively calculable.
  - The KLN theorem is not directly applicable when a sum over initial states is necessary (we have no control on the structure of hadronic initial states).
  - Working at the perturbative, partonic level one identifies sufficiently inclusive cross sections, such that
    - ∗ long-distance effects are suppressed thanks to cancellations (IR–safe cross sections);
    - ∗ long-distance effects can be isolated in universal factors, depending on the initial state but not on the hard process being studied (factorizable cross sections).

The strategy of perturbative QCD

- All calculations are performed at partonic level, with an infrared regulator ($e.\ g.$: $\epsilon = 2 - d/2 < 0$), requiring the presence of at least one hard scale Q^2. One computes

$$\sigma_{\rm part} = \sigma_{\rm part} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \left\{ \frac{m^2(\mu^2)}{\mu^2}, \epsilon \right\} \right) \ .$$

- IR–safe quantities are selected, having a finite limit when the IR regulator removed ($\epsilon \to 0$, $m^2(\mu^2) \to 0$).

$$\sigma_{\rm part} = \sigma_{\rm part} \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \{0, 0\} \right) + \mathcal{O} \left(\left\{ \left(\frac{m^2}{\mu^2} \right)^p, \epsilon \right\} \right) \ .$$

- These partonic, inclusive quantities, admitting a perturbative expansion in powers of $\alpha_s(Q^2) \ll 1$, are interpreted as estimates of the corresponding hadronic quantities, valid modulo $\mathcal{O}\left((\Lambda_{QCD}/Q)^p \right)$ corrections.
- With hadrons in the initial state, the goal is constructing factorizable quantities, such that

$$\sigma_{\rm part} = f \left(\frac{m^2}{\mu_F^2} \right) * \widehat{\sigma}_{\rm part} \left(\frac{Q^2}{\mu^2}, \frac{\mu_F^2}{\mu^2} \right) + \mathcal{O} \left(\left(\frac{m^2}{\mu_F^2} \right)^p \right) \ .$$

- Factorization, proved at parton level, is transcribed at hadron level. Distribution functions f are measured, cross sections $\widehat{\sigma}_{\rm part}$ are derived with a perturbative calculation.

An explicit example: $R_{e^+e^-}$

The prototype of IR–safe cross sections is the total annihilation cross section for $e^+e^- \to \mathrm{hadrons}$.

$$\sigma_{\mathrm{tot}}(q^2) = \frac{1}{2q^2} \sum_X \int d\Gamma_X \, \frac{1}{4} \sum_{\mathrm{spin}} |\mathcal{M}(k_1 + k_2 \to X)|^2 \ ,$$

normalized dividing out the total muon-pair production cross section

$$R_{e^+e^-} \equiv \frac{\sigma_{\mathrm{tot}} \left(e^+e^- \to \mathrm{hadrons} \right)}{\sigma_{\mathrm{tot}} \left(e^+e^- \to \mu^+\mu^- \right)}$$

In $d = 4 - 2\epsilon$, to leading order in α,

$$\sigma_{\mathrm{tot}}(q^2) = \frac{1}{2q^2} L_{\mu\nu}(k_1, k_2) H^{\mu\nu}(q^2) \ ,$$

$$L^{\mu\nu}(k_1, k_2) = \frac{e^2 \mu^{2\epsilon}}{q^4} \left(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1 \cdot k_2 g^{\mu\nu} \right) \ ,$$

$$H^{\mu\nu}(q^2) = e^2 \mu^{2\epsilon} q_f^2 \sum_X \langle 0 | J_\mu(0) | X \rangle \langle X | J_\nu(0) | 0 \rangle (2\pi)^d \delta^d(q - p_X) \ .$$

Current conservation implies $q^\mu H_{\mu\nu} = q^\nu H_{\mu\nu} = 0$, so that

$$H^{\mu\nu}(q^2) = \left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) H(q^2) \ .$$

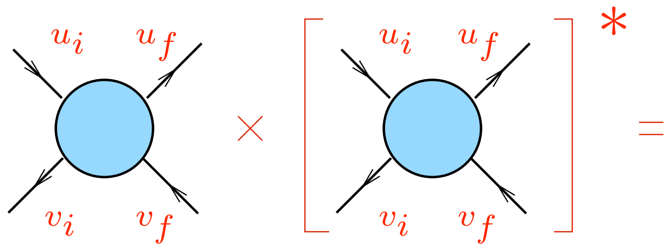
This leads to

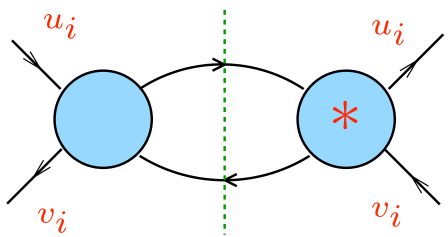
$$-g^{\mu\nu} H_{\mu\nu}(q^2) = (3 - 2\epsilon) \, q^2 H(q^2) \ ,$$

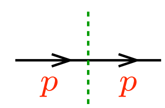
$$\sigma_{\mathrm{tot}}(q^2) = \frac{e^2 \mu^{2\epsilon}}{2q^4} \frac{1 - \epsilon}{3 - 2\epsilon} \left(-g^{\mu\nu} H_{\mu\nu}(q^2) \right) \ .$$

A technical interlude: cut diagrams

A useful representation for $|\mathcal{M}|^2$ can be constructed in terms of cut diagrams. Pictorially







$= \; 2\pi\delta_+(p^2)\; \not{p}$

To the right of the cut all explicit i's in the Feynman rules and all momentum components change sign.
Consistency with spinor and color algebra is easily verified using

$$\left(\overline{\omega}_1 \left[\gamma_{\mu_1}\gamma_{\mu_2}\ldots\gamma_{\mu_i}\gamma_5\ldots\sigma_{\mu\nu}\ldots\gamma_{\mu_n}\right]\omega_2\right)^* =$$
$$\overline{\omega}_2 \left[\gamma_{\mu_n}\ldots\sigma_{\mu\nu}\ldots\gamma_{\mu_i}\gamma_5\ldots\gamma_{\mu_2}\gamma_{\mu_1}\right]\omega_1 \; ,$$

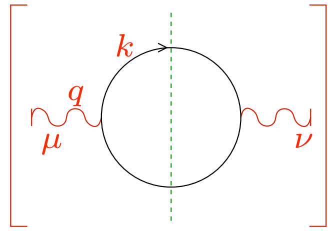
as well as hermiticity of group generators, $\left[(t_a)_{ij}\right]^* = (t_a)_{ji}$.

Note that

- the rules apply for *fixed* final state momenta; the loop momentum integral for cut loops, if needed, becomes the phase space integral;
- for particles with spin $\neq 0$ the cut carries the sum over polarizations;
- cut fermion loops carry the expected minus sign.

Exercise: $R_{e^+e^-}$ at tree–level

$$\sigma_{\rm tot}(q^2) = \frac{e^2\mu^{2\epsilon}}{2q^4}\frac{1-\epsilon}{3-2\epsilon}\left(-g^{\mu\nu}\right)$$



$$-H^\mu_\mu = e^2\mu^{2\epsilon}q_f^2\int\frac{d^dk}{(2\pi)^{d-2}}\delta_+(k^2)\delta_+((k-q)^2){\rm Tr}\left(\not{k}\gamma_\mu(\not{k}-\not{q})\gamma^\mu\right) .$$

In he center of mass frame $(k-q)^2 = q^2 - 2\sqrt{q^2}k_0$; use the two δ_+ distributions to perform k_0 and $|\mathbf{k}|$ integrals; the trace evaluates to $4(1-\epsilon)q^2$.

Summing over quark colors and flavors one finds

$$-H^\mu_\mu = 2(1-\epsilon)e^2\mu^{2\epsilon}N_c\sum_f q_f^2\left(\frac{q^2}{4}\right)^{1-\epsilon}\frac{\Omega_{2-2\epsilon}}{(2\pi)^{2-2\epsilon}} .$$

The d-dimensional solid angle is given by the classic formula

$$\Omega_d = \frac{2^d\pi^{d/2}\Gamma(d/2)}{\Gamma(d)} .$$

In $d = 4-2\epsilon$ one finds then

$$-H^\mu_\mu = 2\ \alpha\ \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\ q^2\left(\frac{4\pi\mu^2}{q^2}\right)^\epsilon N_c\sum_f q_f^2 ,$$

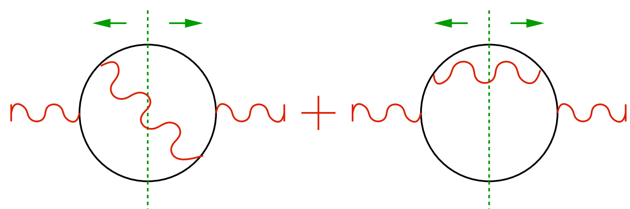
whence the famous result, for $\epsilon \to 0$,

$$\sigma_{\rm tot} = \frac{4\pi\alpha^2}{3q^2}N_c\sum_f q_f^2 \qquad \to \qquad R^{(0)}_{e^+e^-} = N_c\sum_f q_f^2 .$$

Radiative corrections

Enumerating graphs contributing to the one-loop correction is easy in terms of cut diagrams

$$\left[-H^{\mu}_{\mu}\right]^{(1)} =$$



$$+ \ \mathrm{c.t.}$$

Summing over positions of the cuts one obtains real and virtual diagrams in turn. We examine them separately.

Real emission

It is convenient to compute separately the transition probability and three-body space. Following the rules for cut diagrams one finds

$$\left[-H^{\mu}_{\mu}\right]^{(1,R)} = \int \frac{d^d p d^d k}{(2\pi)^{2d-3}} \delta_+(p^2)\delta_+(k^2)\delta_+((p+k-q)^2)\left[-\mathcal{H}^{\mu}_{\mu}\right] \ .$$

The transition probability depends on a single polar angle. Let $u \equiv \cos\theta_{pk}$ in the center of mass frame: one finds

$$\delta_+((p+k-q)^2) = \vartheta\left(p'_0\right)\delta\left(s - 2\sqrt{s}(\hat{p}+\hat{k}) + 2\hat{p}\hat{k}(1-u)\right) \ ,$$

where $\hat{p} = |\mathbf{p}|$, $\hat{k} = |\mathbf{k}|$. One can integrate over the energies of p and k using the respective δ_+. All angular integrations are trivial except the one on u.

Introduce dimensionless variables (quark and gluon energy fractions)

$$z = \frac{2\hat{k}}{\sqrt{s}} \quad , \quad x = \frac{2\hat{p}}{\sqrt{s}} \, ,$$

and define $y = (1-u)/2$. The result is

$$\left[-H^{\mu}_{\mu}\right]^{(1,R)} = \frac{1}{8}\frac{\Omega_{2-2\epsilon}\Omega_{1-2\epsilon}}{(2\pi)^{5-2\epsilon}}\left(\frac{s}{2}\right)^{1-2\epsilon}\int_0^1 dx x^{1-2\epsilon}\int_0^1 dz z^{1-2\epsilon}$$
$$\times \int_0^1 dy\,[y(1-y)]^{-\epsilon}\frac{1}{1-yz}\delta\left(x - \frac{1-z}{1-yz}\right)\left[-\mathcal{H}^{\mu}_{\mu}\right].$$

The transition probability can be computed from the Feynman rules.

$$-\mathcal{H}^{\mu}_{\mu} = -2e^2\mu^{2\epsilon}\sum_f q_f^2 g^2 \mu^{2\epsilon}\mathrm{Tr}(t_a t^a)\left(\frac{\mathrm{Tr}\left[\gamma_\mu(\not{p}+\not{k})\gamma_\sigma\not{p}\gamma^\mu(-\not{p}'-\not{k})\gamma^\sigma\not{p}'\right]}{(2p\cdot k)(2p'\cdot k)}\right.$$
$$\left. + \frac{\mathrm{Tr}\left[\gamma_\mu(\not{p}+\not{k})\gamma_\sigma\not{p}\gamma^\sigma(\not{p}+\not{k})\gamma^\mu\not{p}'\right]}{(2p\cdot k)^2}\right) ,$$

and can be simplified using $\mathrm{Tr}(t_a t^a) = N_c C_F$, Clifford algebra identities such as

$$\begin{aligned}
\gamma_\mu \not{p} \gamma^\mu &= -2(1-\epsilon)\not{p} \, , \\
\gamma_\mu \not{p}\not{k} \gamma^\mu &= 4p\cdot k - 2\epsilon\not{p}\not{k} \, , \\
\gamma_\mu \not{p}\not{k}\not{q} \gamma^\mu &= -2\not{q}\not{k}\not{p} + 2\epsilon\not{p}\not{k}\not{q} \, ,
\end{aligned}$$

and with the identifications $p{\cdot}q = sx/2$, $p{\cdot}k = sxyz/2$, $k{\cdot}q = sz/2$.

One can integrate out the quark energy fraction x, using the remaining δ. The expression simplifies considerably and one gets

$$\left[-H_\mu^\mu\right]^{(1,R)} = 2N_c C_F \sum_f q_f^2 \alpha \alpha_s (1-\epsilon) \frac{\Omega_{2-2\epsilon}\Omega_{1-2\epsilon}}{(2\pi)^{3-4\epsilon}}\, q^2 \left(\frac{2\mu^2}{q^2}\right)^{2\epsilon}$$

$$\int_0^1 dz dy \left[(1-\epsilon)(1-z)^{-2\epsilon}(1-yz)^{-2-2\epsilon} z^{1-2\epsilon}(1-y)^{1-\epsilon}\frac{1}{y^{1+\epsilon}} + \right.$$

$$\left.(1-z)^{1-2\epsilon}(1-yz)^{-2-2\epsilon} z^{1-2\epsilon}\left[y(1-y)\right]^{-\epsilon}\left(\frac{(1-yz)^2}{yz^2(1-y)} - \epsilon\right)\right].$$

One recognizes the announced singularities

- Infrared: $z^{-1-2\epsilon}$, gives a pole in ϵ when the gluon energy z tends to 0.
- Collinear: $y^{-1-\epsilon}$ e $(1-y)^{-1-\epsilon}$, singular when $y \to 0$ (gluon collinear to the quark), and $y \to 1$ (gluon collinear to the antiquark).

Note: mass singularities are regulated choosing $\epsilon < 0$.

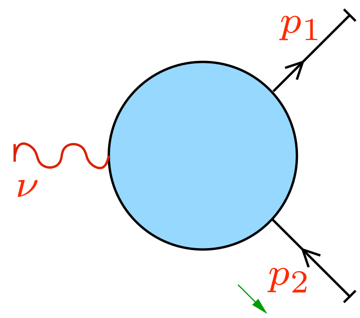
The y and z integrations yield Euler B functions (typical of one-loop calculations in one-scale problems). Expanding around $\epsilon = 0$ one gets the final result for real emission, displaying the double infrared–collinear pole,

$$\left[-H_\mu^\mu\right]^{(1,R)} = N_c C_F \alpha \sum_f q_f^2 \frac{\alpha_s}{\pi}\, q^2 \left(\frac{4\pi\mu^2}{q^2}\right)^{2\epsilon}$$

$$\times \frac{1-\epsilon}{\Gamma(2-2\epsilon)}\left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \pi^2 + \frac{19}{2} + \mathcal{O}(\epsilon)\right].$$

Virtual contribution

Purely virtual contributions to the production amplitude are given to all orders by the quark form factor.

$\Gamma_\nu \left(p_1, p_2; \mu^2, \epsilon\right) =$ 

The calculation greatly simplifies by taking into account the general properties of the form factor.

- For massles quarks, the form factor is expressed in terms of a single scalar function, multiplying the Dirac structure of the threee amplitude.

$$\begin{aligned}\Gamma_\mu(p_1, p_2; \mu^2, \epsilon) &\equiv \langle p_1, p_2 | J_\mu(0) | 0 \rangle \\ &= -ieq_f \, \overline{u}(p_1)\gamma_\mu v(p_2) \, \Gamma\left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) \ .\end{aligned}$$

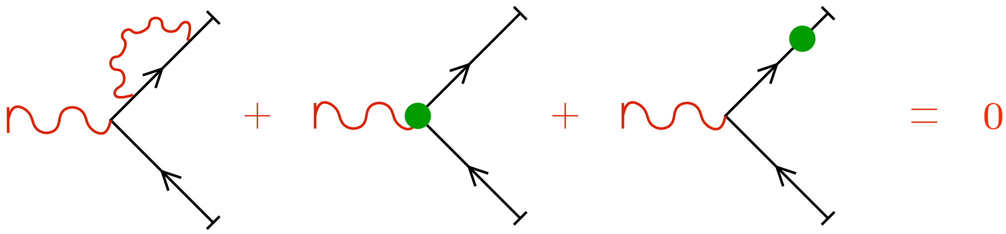
  As a consequence, the transition probability is proportional to the tree–level result, with an overall factor given by $2 \, \mathrm{Re}\Gamma$.
- The form factor is renormalization group invariant (it has a vanishing anomalous dimension), as a consequence of the conservation of the electromagnetic current.

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\epsilon, \alpha_s)\frac{\partial}{\partial\alpha_s}\right)\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) = 0 \ .$$

  QCD does *not* violate the QED Ward identity. $Z_1 = Z_\psi$.

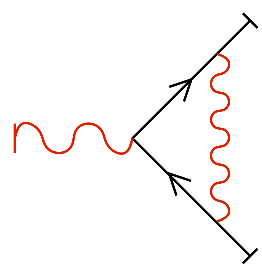
- Reducible (1PR) graphs on each fermion line, including the respective counterterms, reconstruct the residue R_ψ of the quark propagator. Since, according to the reduction formulas, every external line must be multiplied times $R_\psi^{-1/2}$, it is necessary to include these graphs on one of the two lines only.
- In Feynman gauge and in dimensional regularisation all 1PR graphs with the loop on the external fermion line vanish because they are expressed by scale-less integrals ($p_i^2 = 0$).
  - Note! This is false in general in axial gauges ($\exists\ n \cdot p_i$); furthermore it depends on a cancellation of IR and UV effects ...

At one loop these observations are summarized by the identity



One is left with only one graph to be computed, the vertex correction

$$\Gamma_\nu^{(1)}\left(p_1, p_2; \mu^2, \epsilon\right) =$$



which can be written as

$$\Gamma_\nu^{(1)} = -eq_f g^2 \mu^{2\epsilon} C_F \int \frac{d^d k}{(2\pi)^d} \frac{\overline{u}(p_1)\gamma_\sigma(\not{p}_1 - \not{k})\gamma_\nu(\not{p}_2 + \not{k})\gamma^\sigma v(p_2)}{k^2(p_1 - k)^2(p_2 + k)^2}.$$

The steps to compute this diagram are standard. Summarizing

- Sistematically use the mass–shell conditions (Dirac equation), $\overline{u}(p_1)\not{p}_1 = \not{p}_2 v(p_2) = 0$, then isolate integrals with different powers of k.

$$\Gamma_\nu^{(1)} = -e q_f g^2 \mu^{2\epsilon} C_F \overline{u}(p_1) \Big[2q^2 \gamma_\nu I_0 + 2(\gamma_\nu \gamma_\alpha \not{p}_1 - \not{p}_2 \gamma_\alpha \gamma_\nu) I^\alpha - \gamma^\sigma \gamma_\alpha \gamma_\nu \gamma_\beta \gamma_\sigma I^{\alpha\beta} \Big] v(p_2) \, .$$

- The tensor integrals

$$I_0 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (p_1 - k)^2 (p_2 + k)^2} \, ,$$

$$I_\alpha = \int \frac{d^d k}{(2\pi)^d} \frac{k_\alpha}{k^2 (p_1 - k)^2 (p_2 + k)^2} \, ,$$

$$I_{\alpha\beta} = \int \frac{d^d k}{(2\pi)^d} \frac{k_\alpha k_\beta}{k^2 (p_1 - k)^2 (p_2 + k)^2} \, ,$$

  can be computed with the usual Feynman parametrization.

  – Note: only I_0 can have IR divergences, and only $I_{\alpha\beta}$ can have UV divergences. I_α may have collinear poles ...

- It may be useful to decompose tensor integrals *à la* Passarino–Veltman, in order to get directly the scalar form factor Γ. The result is

$$I^\alpha = p_1^\alpha \, I_1 + p_2^\alpha \, I_2 \, ,$$

$$I^{\alpha\beta} = g^{\alpha\beta} \, I_3 + p_1^\alpha p_1^\beta \, I_4 + p_2^\alpha p_2^\beta \, I_5 + (p_1^\alpha p_2^\beta + p_1^\beta p_2^\alpha) \, I_6 \, .$$

- In terms of the scalar integrals $I_1, \ldots, I_6$ one finds

$$\Gamma^{(1)} = g^2 \mu^{2\epsilon} C_F \left[4(1-\epsilon)^2 I_3 - 2q^2 \left(I_0 + I_2 - I_1 + (1-\epsilon) I_6 \right) \right] .$$

- The final result for the form factor is

$$\Gamma^{(1)} = -\frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-q^2} \right)^{\epsilon} \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 + \mathcal{O}(\epsilon) \right] .$$

When taking the real part one must use

$$(-q^2 + \mathrm{i}\varepsilon)^{-\epsilon} = (q^2)^{-\epsilon} \, \mathrm{e}^{-\mathrm{i}\pi\epsilon} .$$

Note! The sign of q^2 is dictated by the Cutkosky rules. Because of the double pole, the factor $\exp(-\mathrm{i}\pi\epsilon)$ must be expanded to second order in ϵ. It generates numerically important contributions.

Result

We observe that, as expected, the virtual contribution has the same IR-C poles as real emission. Summing the two, the poles cancel and one can take the limit $\epsilon \to 0$, with the result

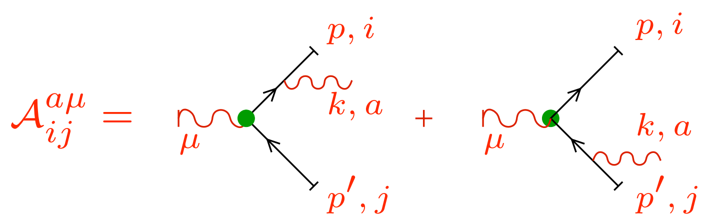
$$\sigma_{\mathrm{tot}} = \frac{4\pi\alpha^2}{3q^2} N_c \sum_f q_f^2 \left(1 + \frac{\alpha_s}{\pi} \frac{3}{4} C_F + \mathcal{O}(\alpha_s^2) \right) ,$$

For $SU(3)$, where $C_F = 4/3$, the (classical) result is

$$R^{(0)}_{e^+e^-} = N_c \sum_f q_f^2 \left(1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) \right) .$$

Soft approximation

The cancellation that was just exhibited is possible only because, in the IR and C limits the amplitude for the emission of a real gluon becomes proportional to the Born amplitude, just as was the case for the virtual diagram. This result can be made systematic introducing the soft approximation.



$$\mathcal{A}_{ij}^{a\mu} = g t_{ij}^a \overline{u}(p) \left[\frac{\not\varepsilon(k)(\not p + \not k)\Gamma_\mu}{2p \cdot k} - \frac{\Gamma_\mu(\not p' + \not k)\not\varepsilon(k)}{2p' \cdot k} \right] v(p') \ .$$

When the gluon is soft one can

- neglect k in the numerator, and in the definition of p';
- commute $\not p$ and $\not p'$ in order to impose the mass-shell condition, $\not p' v(p') = \overline{u}(p)\not p = 0$.

The result is

$$\mathcal{A}_{ij}^{a\mu}\Big|_{\mathrm{soft}} = g t_{ij}^a \left[\frac{p \cdot \varepsilon}{p \cdot k} - \frac{p' \cdot \varepsilon}{p' \cdot k} \right] \mathcal{A}_0^\mu \ ,$$

where $\mathcal{A}_0^\mu = \overline{u}(p)\Gamma^\mu v(p')$ is the Born amplitude (whatever the explicit form of the vertex Γ_μ).

Remarks

- The soft amplitude is gauge–invariant (it vanishes if $\varepsilon \propto k$).
- Soft gluon emission has universal characters. Long-wavelength gluons cannot analyze the short-distance properties of the emitter (spin, internal structure), they only detect the global color charge and the direction of motion These considerations generalize to multiple emission.
- Identical considerations apply to gluon emission from gluons.

Soft cross section

It's easy to recover the singular part of the real emission cross section. The transition probability can be computed summing over colors and polarizations (using $\sum \varepsilon_\mu \varepsilon_\nu^* = -g_{\mu\nu}$, which is allowed in this case).

$$|\mathcal{A}_{\rm soft}|^2 = g^2 C_F \, |\mathcal{A}_0|^2 \, \frac{2p \cdot p'}{p \cdot k \; p' \cdot k} \, .$$

To get the cross section one integrates over phase space

$$\sigma_{q\bar{q}g}^{\rm soft} = g^2 C_F \sigma_{q\bar{q}} \int \frac{d^3k}{2|\mathbf{k}|(2\pi)^3} \frac{2p \cdot p'}{p \cdot k \; p' \cdot k} \, .$$

In the center of mass frame ($\mathbf{q} = \mathbf{0}$) and in the soft approximation the quark and the antiquark are still back to back. One then recovers the structure of IR and C singularities,

$$\sigma_{q\bar{q}g}^{\rm soft} = \sigma_{q\bar{q}} C_F \frac{\alpha_s}{\pi} \int_{-1}^{1} d\cos\theta_{pk} \int_0^\infty \frac{d|\mathbf{k}|}{|\mathbf{k}|} \frac{2}{(1-\cos\theta_{pk})(1+\cos\theta_{pk})} .$$

Virtual diagrams

The soft approximation can be applied to virtual diagrams as well, with some care.

- When $k_\mu \ll \sqrt{q^2}$, $\forall \mu$, one can neglect k^2 with respect to $p_i \cdot k$ in denominators, as well as k in numerators (eikonal approximation).
  - Note: the approximation is not uniformly valid, in some cases it becomes necessary to deform integration contours, or the approximation may break down.
- Using light-cone coordinates one can set

$$p^\mu = \left(p^+, 0, \mathbf{0}_\perp\right) \;, \qquad (p')^\mu = \left(0, (p')^-, \mathbf{0}_\perp\right) \;.$$

  - Note: For a generic four-vector, $v^\mu = \left(v^+, v^-, \mathbf{v}_\perp\right)$, $v^\pm = (v^0 \pm v^3)/\sqrt{2}$, $v^2 = 2v^+v^- - |\mathbf{v}_\perp|^2$.

- Consider for example the integral I_0, containing the virtual double pole. In the eikonal approximation and in $d = 4$

$$I_0^{(\mathrm{eik})} = \frac{1}{32\pi^4 q^2} \int \frac{dk^+ dk^- d^2k_\perp}{(-k^- + \mathrm{i}\epsilon)(k^+ + \mathrm{i}\epsilon)(2k^+k^- - |\mathbf{k}_\perp|^2 + \mathrm{i}\epsilon)} \;.$$

  There are three integration regions giving rise to divergences. One can parametrize them introducing a scaling variable λ, according to

$$k^\mu \sim \lambda \sqrt{q^2} \;, \; \forall \mu \;, \qquad \rightarrow \quad \mathrm{IR} \;;$$

$$k^\pm \sim \sqrt{q^2} \;, k^\mp \sim \lambda^2 \sqrt{q^2} \;, |\mathbf{k}_\perp| \sim \lambda \sqrt{q^2} \;, \qquad \rightarrow \quad \mathrm{COLL} \;.$$

Angular ordering

The soft approximation is of great practical relevance in perturbative QCD.

- It displays the universal properties of soft color radiation (color transparency, angular ordering)
- It links perturbative and non–perturbative regimes (resummations, hadronization Monte Carlo)

The simplest example of angular ordering is the differenzial cross section $d\sigma_{q\bar{q}g}$ in a frame in which the decaying photon has a large momentum $\mathbf{q}$. In that frame the quark and the antiquark are typically emitted forming a small angle ($\theta_{pp'} \ll \pi$), and one has

$$\begin{aligned} d\sigma^{\rm soft}_{q\bar{q}g} &= d\sigma_{q\bar{q}} C_F \frac{\alpha_s}{\pi} \frac{d|\mathbf{k}|}{|\mathbf{k}|} d\cos\theta_k \frac{d\phi_k}{2\pi} \frac{1-\cos\theta_{pp'}}{(1-\cos\theta_{pk})(1-\cos\theta_{p'k})} \\ &= d\sigma_{q\bar{q}} C_F \frac{\alpha_s}{\pi} \frac{d|\mathbf{k}|}{|\mathbf{k}|} d\cos\theta_k \frac{d\phi_k}{2\pi} \frac{1}{2} \left(W_q + W_{\bar{q}} \right) \end{aligned}$$

where

$$W_q = \frac{1-\cos\theta_{pp'}}{(1-\cos\theta_{pk})(1-\cos\theta_{p'k})} + \frac{1}{(1-\cos\theta_{pk})} - \frac{1}{(1-\cos\theta_{p'k})} \, .$$

while $W_{\bar{q}}$ is found exchanging $p \leftrightarrow p'$.
The full angular distribution is positive definite, but singular for emissions parallel to either the quark or the antiquark. The partial distributions W_i are not positive definite, but they enjoy special properties.

Properties of W_q e $W_{\bar{q}}$

- W_q is singular only when $\cos\theta_{pk} \to 1$, while the opposite is true for $W_{\bar{q}}$.
- The azimuthal average of W_q (with respect to the axis defined by $\mathbf{p}$) vanishes if $\theta_{pk} > \theta_{pp'}$.

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi \; W_q(\phi) = \frac{2}{1-\cos\theta_{pk}} \, \Theta\left(\theta_{pp'} - \theta_{pk}\right) \; ,$$

as can be proven using

$$\cos\theta_{p'k} = \cos\theta_{pk}\cos\theta_{pp'} + \sin\theta_{pk}\sin\theta_{pp'}\cos\phi \; .$$

An identical equation applies to $W_{\bar{q}}$.

- Azimuthal averages are positive definite and can be interpreted as probability distributions for the emission of soft gluons independently of the quark and of the antiquark.

Thus: the distribution of soft radiation on average is given by a sum of uncorrelated contributions from the quark and from the antiquark, each vanishing outside the cones built rotating the direction of one of the fermions around that of the other one.

Comments

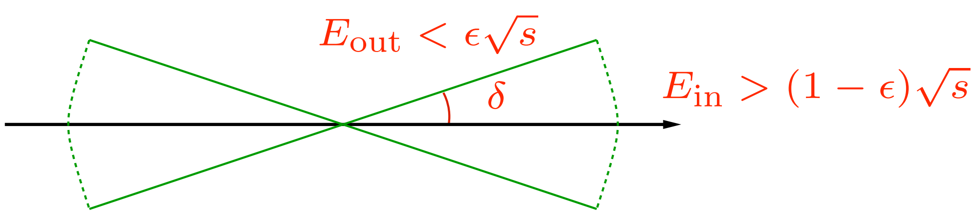
- This property can be generalized to higher orders. Gluons which are radiated later are forced on average to the inside of the cones defined by previously emitted gluons and quarks.
- Perturbative evolution in the soft limit is local in phase space, so that color singlet parton clusters are likely to be formed inside collimated particle beams (jets).

Sterman–Weinberg jets

Momentum configurations responsible for singularities and needed for their cancellation are infrared and collinear, as expected.

Thus it is not necessary to integrate real emission over the entire phase space in order to obtain a finite result, it is sufficient to consider sufficiently inclusive observables, such that gluon emission be integrated over IR and C configurations.

Prototype: two-jet cross section



Definition: an event is a two-jet event iff $\exists$ two opposite cones with opening angle δ, such that all the energy, except at most a fraction ϵ, flows into the cones.

At parton level:

- All events are two-jet events at leading order.
- At $\mathcal{O}(\alpha_s)$ two-jet events are those in which the gluon is IR (and emitted in any direction), or C (with any energy). All other events are three-jet events.
- Virtual contributions are two-jet events. Therefore the partonic two-jet cross section is finite.

More precisely:

- At LO one finds simply $\sigma_{2\mathrm{j}}^{(0)}(\epsilon, \delta) = \sigma_{\mathrm{tot}}^{(0)} = N_c \sum_f q_f^2 \frac{4\pi\alpha^2}{3q^2}$.
- At NLO one finds only two- or three-jet events, so that

$$\sigma_{2\mathrm{j}}^{(1)}(\epsilon, \delta) = \sigma_{\mathrm{tot}}^{(1)} - \sigma_{3\mathrm{j}}^{(1)}(\epsilon, \delta) \ ,$$

- $\sigma_{3\mathrm{j}}^{(1)}$ is easily computed from the real emission matrix element with appropriate cuts on phase space. In $d = 4$

$$\begin{aligned} \left[-H_\mu^\mu\right]_{3\mathrm{j}}^{(1,R)} &= 2N_c C_F \sum_f q_f^2 \alpha \frac{\alpha_s}{\pi} q^2 \int_{2\epsilon}^1 dz \\ &\times \int_{\delta^2}^{1-\delta^2(1-z^2/2)} dy \left[\frac{z(1-y)}{y(1-yz)^2} + \frac{1-z}{yz(1-y)}\right] \ . \end{aligned}$$

- It is easy to compute the dominant contributions as $\epsilon, \delta \to 0$. Combining with the leptonic tensor one finds

$$\sigma_{3\mathrm{j}}^{(1)}(\epsilon, \delta) = \sigma_{\mathrm{tot}}^{(0)} C_F \frac{\alpha_s}{\pi} \left[4\log(\delta)\log(2\epsilon) + 3\log(\delta) + \frac{\pi^2}{3} - \frac{7}{4}\right] \ .$$

Observe:

- The total cross section is dominated by two-jet events at large q^2 (asymptotic freedom for jets ...).
- For increasing q^2 the perturbative result remains reliable for narrower cones, jets are more collimated.
- It is possible to compute (and verify experimentally) the angular distribution of two-jet events $d\sigma_{2j}/d\cos\theta \propto 1 + \cos^2\theta$, as expected for spin $1/2$ quarks.

Event–shape variables

The mechanism underlying the cancellation of IR and C divergences suggests a further generalization: study distributions of observables constructed with final state momenta so as to assign equal weights to events differing by IR or C emissions.

Given a final state with m partons, let $E_m(p_1, \ldots, p_m)$ be the observable. The corresponding distribution is defined by

$$\frac{d\sigma}{de} = \frac{1}{2q^2} \sum_m \int d\mathrm{LIPS}_m \, \overline{|\mathcal{M}_m|^2} \, \delta\left(e - E_m(p_1, \ldots, p_m)\right) \; ,$$

and its moments (and in particular the average value) are

$$\langle e^n \rangle = \int_{e_{\min}}^{e_{\max}} de \; e^n \frac{d\sigma}{de} \; .$$

Note: These are "weighted" cross sections.
At order α_s^{m-1} one must sum contributions with $m+1$ partons in the final state, with one virtual m real partons, and so on.

$$\sigma(e) \Big|_{\mathcal{O}\left(\alpha_s^{m+1}\right)} = \int d\sigma_{m+1}^{(R)} + \int d\sigma_m^{(1V)} + \ldots \; .$$

IR-C cancellation is preserved if the observable takes the same value for those configurations differing only by the IR/C radiation.

$$\lim_{p_j^\mu \to 0} E_{m+1}(p_1, \ldots, p_j, \ldots) = E_m(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots) \; ,$$

$$\lim_{p_k^\mu \to \alpha p_j^\mu} E_{m+1}(p_1, \ldots, p_j, \ldots, p_k, \ldots) = E_m(p_1, \ldots, p_j + p_k, \ldots) \; .$$

Examples of event–shape variables

There is a variety of available IR/C safe event shapes.

- Thrust

$$T_m = \max_{\hat{\mathbf{n}}} \frac{\sum_{i=1}^m |\mathbf{p_i} \cdot \hat{\mathbf{n}}|}{\sum_{i=1}^m |\mathbf{p_i}|} \, .$$

  Clearly $0 < T_m \leq 1$, and $T_m = 1$ corresponds to two precisely collimated back to back particle beams.

- C parameter

$$C_m = 3 - \frac{3}{2} \sum_{i,j=1}^m \frac{\left(p_i \cdot p_j\right)^2}{\left(p_i \cdot q\right)\left(p_j \cdot q\right)} \, .$$

  Also in this case $0 \leq C_m \leq 1$; two-jet eventi have $C = 0$. The definition can be expressed in terms of the eigenvalues of the space part of the energy-momentum tensor of the final state, $C = 3(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$.

- Jet masses

$$\rho_m^{(H)} = \frac{1}{q^2} \left(\sum_{p_i \in H} p_i \right)^2$$

  H is one of the two hemispheres identified by the thrust axis.

Observe

- Perturbative distributions are singular in the two-jet limit (logarithms of the form $\alpha_s^n \log^{2n-1} C$), but expectation values are finite.
- Great phenomenological relevance (for example: determination of α_s, hadronization corrections).
- Jet algorithms and the related definitions of multi-jet events can be seen as particular event-shapes.

A comparison with QED

QED also has IR divergences, as well as collinear divergences in the massless limit. There are similarities and differences.

- In QED, consider for example the process $e^+e^- \to \mu^+\mu^-$.
  - $\sigma(e^+e^- \to \mu^+\mu^-$ diverges starting at $\mathcal{O}(\alpha^3)$.
  - Therefore $\sigma_{\rm Born}$ is not a good approximation for σ. This is not a problem. The true observable is $\sigma_{\rm tot}(\Delta) = \sum_n \sigma(e^+e^- \to \mu^+\mu^- + n\gamma, \Delta)$, and for $\sigma_{\rm tot}(\Delta)$, which is finite, $\sigma_{\rm Born}$ is a good approximation.
  - IR divergences in QED can be explicitly resummed

    $$\sigma_{\rm tot}(\Delta) = \sigma_{\rm fin} \exp\left[\frac{\alpha}{\pi} \log\left(\frac{\Delta^2}{q^2}\right) f(m^2, q^2)\right] .$$

    so that $\lim_{\Delta \to 0} \sigma_{\rm tot}(\Delta) = 0$.
  - Interpretation: it is not possibile to produce only $\mu^+\mu^-$; asimptotic states of QED are not isolated fermions.
- In QCD, considering $e^+e^- \to q\bar{q}$, the situation is almost analogous.
  - $\sigma(e^+e^- \to q\bar{q}$ diverges starting at $\mathcal{O}(\alpha^2\alpha_s)$.
  - Therefore $\sigma_{\rm Born}$ is not a good approximation for $\sigma = 0$ (confinement). On the other hand it is a good approximation for $\sigma_{\rm tot}(e^+e^- \to {\rm hadrons})$, which is finite.
  - Interpretation: it is not possible to produce only $q\bar{q}$; the asimptotic states of QCD are not quarks and gluons.

On singularities of Feynman diagrams

To study IR and C divergences to all orders one must characterize the generic singularity structure of Feynman diagrams. Let us begin with a simple example, I_0.

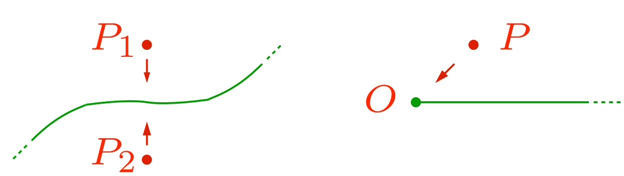
Example: the scalar form factor

Introduce Feynman parameters y_1, y_2, y_3. Then

$$I_0 = 2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 \prod_{i=1}^{3} dy_i \frac{\delta(1 - y_1 - y_2 - y_3)}{\left[y_1 k^2 + y_2 (p - k)^2 + y_3 (p' + k)^2 + \mathrm{i}\epsilon\right]}.$$

Let D_0 be the denominator. The only possible singularities of I_0 must lie on surfaces where $D_0 = 0$. Furthermore

- The integrand is a function of the complex variables k^μ, y_i, with an analytic structure determined by the $\mathrm{i}\epsilon$ prescription.
- The vanishing of D_0 on one integration contour is not sufficient to determine a singularity of the integral. The contour can be deformed.
- Only in two cases the singularity cannot be avoided by a deformation: either the contour is trapped between two poles, or one has end-point singularities, a pole migrates to the edge of an integration contour.



- Singularities of I_0
  - dk^μ integrals cannot have end-point singularities (I_0 is UV convergent). D_0 however is quadratic in k^μ, so that two poles can trap the contour if they coalesce,

$$\frac{\partial}{\partial k^\mu} D_0 \left(y_i, k^\mu, p, p' \right) = 0 \, .$$

  - dy_i integrals can only have end-point singularities (in $y_i = 0$), since D_0 is linear in y_i. Alternatively, D_0 can be independent of y_i on the surface $D_0 = 0$, so that y_i becomes useless for the deformation.

- Landau equations for I_0
  A necessary condition for a singularity of I_0 is that all integration variables be trapped. This is expressed by the Landau equations

$$y_1 k^\mu - y_2 (p - k)^\mu + y_3 (p' + k)^\mu = 0 \qquad \text{and}$$
$$y_i = 0 \qquad \text{or} \qquad l_i^2 = 0 \, ,$$

  where l_i^μ is the momentum flowing through the line with parameter y_i.

- Solutions of Landau equations
  It's easy to find the expected solutions.

$$k^\mu = 0 \, ; \; y_2/y_1 = y_3/y_1 = 0 \qquad \text{IR}$$
$$k^\mu = \alpha p^\mu \, ; \; y_3 = 0 \, ; \; \alpha y_1 = (1 - \alpha) y_2 \qquad \text{C}$$
$$k^\mu = - \beta p^\mu \, ; \; y_2 = 0 \, ; \; \beta y_1 = (1 - \beta) y_3 \qquad \text{C}$$

  We recognize the expected IR and C singularities. Are there other solutions of Landau equations?

- Coleman–Norton physical picture

  The search for solutions of Landau equations is simplified by the fact that they admit a simple physical representation. Observe that

  – if line i in a loop is off–shell it must be that $y_i = 0$.

  – Let $\Delta x_i^\mu \equiv y_i l_i^\mu$, for each on-shell line l_i. Then

  $$\Delta x_i^\mu = \Delta x_i^0 \, v_i^\mu \quad ; \quad v_i^\mu = \left(1, \frac{\mathbf{l}_i}{l_i^0} \right) .$$

  Interpretation: Δx_i^μ describes classical propagation of a massless particle with momentum l_i.

  – The Landau equations for I_0 can now be written as

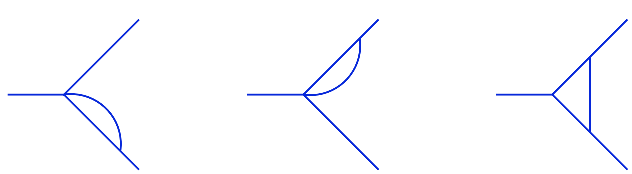
  $$\sum_i \sigma(i) \; \Delta x_i^\mu = 0 \qquad \text{on shell}$$

  $$\Delta x_i^\mu = 0 \qquad \text{off shell} .$$

- Interpretation: the solutions of Landau equations are given by reduced diagrams, where

  – Off–shell lines are contracted to points.

  – On–shell lines describe physically admissible processes for the classical propagation of massless particles.



General case

- Feynman parametrization

  Thanks to the identity

$$\prod_{i=1}^{N} \frac{1}{D_i^{a_i}} = \frac{\Gamma\left(\sum_{i=1}^{N} a_i\right)}{\prod_{i=1}^{N} \Gamma(a_i)} \int_0^1 \prod_{i=1}^{N} \left(dy_i y_i^{a_i-1}\right) \frac{\delta\left(1-\sum_{i=1}^{N} y_i\right)}{\left(\sum_{i=1}^{N} y_i D_i\right)^{\sum_{i=1}^{N} a_i}},$$

  an arbitrary Feynman diagram $G(p_r)$ can be written as

$$G(p_i) = \prod_{\text{linee}} \int_0^1 dy_i \delta\left(1 - \sum_i y_i\right) \prod_{\text{loops}} \int d^d k_l \frac{\mathcal{N}\left(y_i, k_l, p_r\right)}{\left[\mathcal{D}\left(y_i, k_l, p_r\right)\right]^N},$$

  where the denominator $\mathcal{D}$ is a sum of propagators

$$\mathcal{D}\left(y_i, k_l, p_r\right) = \sum_{\text{linee}} y_i \left(l_i^2(p,k) - m_i^2\right) + i\epsilon ,$$

  and the momenta l_i are linear functions of p_r and k_l.

- Landau equations

$$\sum_i \eta_{ij}\, \Delta x_i^\mu = 0 \qquad \text{on shell} ,\ \forall j/i \in j ,$$

$$\Delta x_i^\mu = 0 \qquad \text{off shell} .$$

- Coleman–Norton picture

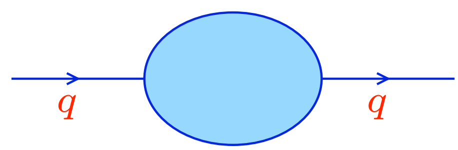
  Solutions are again reduced diagrams (with off–shell lines contracted to points), where all remaining loops can be interpreted as classically permitted processes. For any loop, it must be possible to associate to all its vertices coordinates x_k^μ, $k = 1, \ldots, M$ such that

$$\Delta x_{12}^\mu + \ldots + \Delta x_{M1}^\mu = 0 ,\ \Delta x_{ij}^\mu \equiv x_i^\mu - x_j^\mu .$$

Example: the two-point function

Consider an arbitrary 1PI diagram for the two-point function $G(q^2, m^2)$, in a theory with only one species of particle with mass m^2.

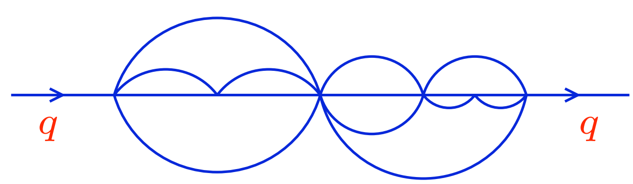
$$G(q^2, m^2) =$$



Theorem: the only singularities of the diagram (and thus of $G(q^2, m^2)$) are normal thresholds $q^2 = n^2 m^2$, $n = 1, 2 \ldots$.

Proof:

- Normal thresholds are solutions of Landau equations.
  In fact, when $q^2 > 0$ one can choose $q^\mu = \left(\sqrt{q^2}, \mathbf{0}\right)$. The Coleman–Norton process is the creation of $n > 1$ particles at rest, which do not move, and interact until they are absorbed, for an arbitrarily long time. An example with $n = 4$ is



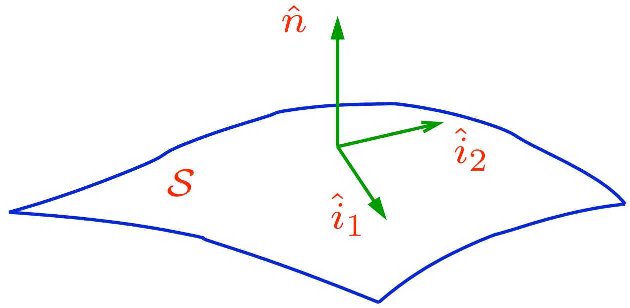
- No other reduced diagram satisfies Coleman–Norton.
  If one produced particle has non-vanishing momentum the other ones must compensate moving in the opposite direction. Once separated they cannot meet again in free motion.

IR/C power counting

The Landau equations are only necessary conditions for the manifestation of IR/C divergences. If phase space has high enough dimension singularities are suppressed (e.g.: IR divergences in ϕ_6^3).

One must develop power counting techniques, similar to those employed in the UV, to determine the strength of the singularities.

- Given a diagram, use the CN representation to identify a trapped surface $\mathcal{S}$ in the space $\{k_i^\mu, y_i\}$.
- For every $\mathcal{S}$, identify among the $\{k_i^\mu\}$ intrinsic coordinates (movement in $\mathcal{S}$) and normal coordinates (distance from $\mathcal{S}$).



  Example: for I_0, $k \| p$, k^+ is intrinsic, $\{k^-, \mathbf{k}_\perp\}$ are normal.

- Introduce a scaling variable determine the relative weight (integration volume)/singularity. Set $n_i = \lambda^{a_i}\, \hat{n}_i$ and consider $\lambda \to 0$, $\hat{n}_i$ finite.

  Example: for I_0, $k \| p$, $k^- \sim \lambda^2 \sqrt{q^2}$, $|\mathbf{k}_\perp| \sim \lambda \sqrt{q^2}$.

- Construct the homogeneous integral for $\mathcal{S}$, taking the dominant power of λ in every factor of the graph.

  Example: for I_0, the homogeneous integral is the eikonal one.

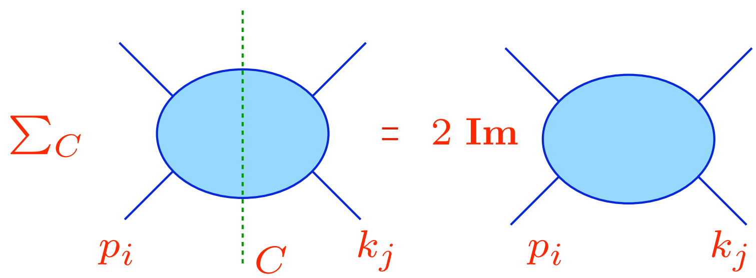
- The degree of divergence is given by the power of λ associated with the homogeneous integral. If, for every line, $l_i^2(p,k) - m_i^2 \to \lambda^{A_i} f(\hat{n})$, then

$$n_{\mathcal{S}} = \sum_i a_i - \sum_i A_i + n_{\rm num} \, .$$

$n_{\mathcal{S}} \leq 0$ signals a divergence, logarithmic when $n_{\mathcal{S}} = 0$.

Application: finiteness of $R_{e^+e^-}$

The all-order finiteness of $\sigma_{\rm tot}(e^+e^- \to {\rm hadrons})$ follows from its relationship with the correlator of two elettromagnetic currents, which in turn comes from unitarity,



generalizing $TT^\dagger = -{\rm i}(T - T^\dagger)$. Define then

$$\rho_{\mu\nu}(q) \equiv {\rm i}e^2 \int d^4x {\rm e}^{{\rm i}qx} \langle 0 | T \left[J_\mu(x) J_\nu(0) \right] |0\rangle$$
$$= \left(q_\mu q_\nu - q^2 g_{\mu\nu} \right) \pi(q^2) \, .$$

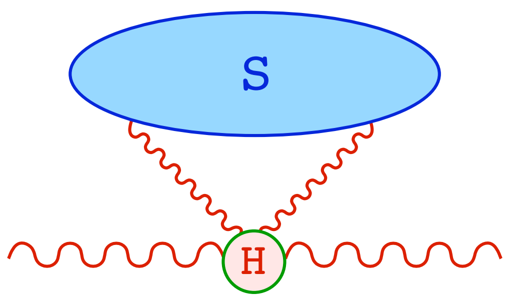
Unitarity gives

$$2 \, {\bf Im} \left[\rho_{\mu\nu}(q) \right] = e^2 \sum_n \langle 0 | J_\mu(0) | n \rangle \langle n | J_\nu(0) | 0 \rangle (2\pi)^4 \delta^4 \left(q - p_n \right) \, ,$$

$$\sigma_{\rm tot}(e^+e^- \to {\rm hadrons}) = \frac{e^2}{q^2}\,{\bf Im}\left[\pi(q^2)\right] \ .$$

It is then sufficient to prove the finiteness of $\pi(q^2)$. This follows from the Coleman-Norton representation.

- In a frame in which $q^\mu = \left(\sqrt{q^2}, \mathbf{0}\right)$ one sees that there are no allowed classical processes with non-vanishing momenta such that the photon decays and then reconstitutes. Thus there are no trapped surfaces with non-vanishing momenta.
- The only trapped surfaces are those with all particles having vanishing momentum, and reduced diagram



These however are finite by power counting, as may be expected (all lines in H are off–shell).

- To see it note that fermion lines at zero momentum are less singular than gluon lines. The worst case is then if $\mathcal{S}$ contains only gluons. In that case, in d dimensions, with $L_\mathcal{S}$ loops and $g_\mathcal{S}$ gluons in $\mathcal{S}$,

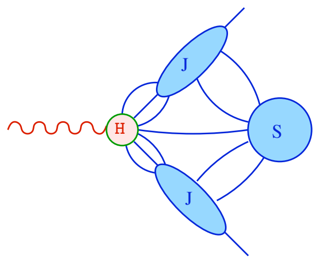
$$n_\mathcal{S} = d\ L_\mathcal{S} - 2g_\mathcal{S} = 2(1-\epsilon)g_\mathcal{S}$$

which is positive in $d > 2$.

Application: the quark form factor

Considering $\Gamma(q^2)$ one finds collinear divergences associated with observed quarks. The most general reduced diagram is

$$\Gamma_\nu \left(p_1, p_2; \mu^2, \epsilon\right) =$$



Further simplifications are possible

- Gluons connecting $\mathcal{S}$ directly to H are suppressed (one more off–shell propagator, plus a new soft propagator dominated by the new soft loop).
- Fermion lines connecting different subgraphs (except the necessary q e $\bar{q}$) are suppressed.
- In an axial gauge, $n \cdot A = 0$, only the (anti)quark line connects the jet J to H. In fact the axial gauge gluon propagator is

$$G^{\rm ax}_{\mu\nu}(k) = \frac{1}{k^2 + {\rm i}\varepsilon} \left(-g_{\mu\nu} + \frac{n_\mu k_\nu + n_\nu k_\mu}{n \cdot k} - n^2 \frac{k_\mu k_\nu}{(n \cdot k)^2} \right) .$$

  Contracting with k^μ cancels the denominator

$$k^\mu G^{\rm ax}_{\mu\nu}(k) = \frac{n_\nu}{n \cdot k} - n^2 \frac{k_\nu}{(n \cdot k)^2} ,$$

  so that the degree of IR/C singularity is reduced.
- These considerations suggest a factorization $\Gamma = J_1 J_2 \mathcal{S} H$.

Diagrammatics of factorization

Identifying the leading integration regions in momentum space is only the first step towards factorization. One must then

- exploit simplifications in Feynman diagrams in the leading regions (soft/collinear approximations, Ward identities).
- organize all-order subtractions a tutti gli ordini so as to avoid double counting (different factor functions have operator definitions, in general non local).

Microexample: again the one-loop form factor, collinear region $k \| p_1$, Feynman gauge.

- Kinematics: $p_1^\mu = \left(p_1^+, 0, \mathbf{0}_\perp\right)$, $k^\mu = \left(k^+, k^-, \mathbf{k}_\perp\right)$, con $k^+ \gg \{k^-, \mathbf{k}_\perp\}$.
- Grammer-Yennie approximation in the numerator

$$\overline{u}(p_1)\gamma_\sigma(\not{p}_1 - \not{k})\gamma_\mu(\not{p}_2 + \not{k})\gamma^\sigma v(p_2) \rightarrow \overline{u}(p_1)\gamma^+(\not{p}_1 - \not{k})\gamma_\mu(\not{p}_2 + \not{k})\gamma^- v(p_2)$$
$$\rightarrow \quad \frac{1}{k^+}\overline{u}(p_1)\gamma^+(\not{p}_1 - \not{k})\gamma_\mu(\not{p}_2 + \not{k})k^+\gamma^- v(p_2) \quad \rightarrow$$
$$\rightarrow \quad \frac{1}{k \cdot \hat{u}_2}\overline{u}(p_1)\gamma^+(\not{p}_1 - \not{k})\gamma_\mu(\not{p}_2 + \not{k}) \; \not{k} \; v(p_2) \; .$$

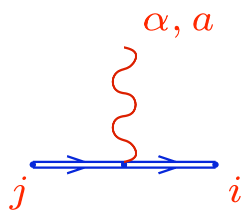
- Ward identity: $\not{k} = (\not{p}_2 + \not{k}) - \not{p}_2$. One finds

$$\Gamma_\mu^{(\mathrm{coll})} \propto \int d^d k \, \frac{\overline{u}(p_1) \; \not{u}_2 \; (\not{p}_1 - \not{k}) \; \gamma_\mu \; v(p_2)}{k^2 \; (p_1 - k)^2 \; k \cdot u_2} \; .$$

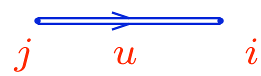
  The gluon-antiquark coupling simplifies, recognizing only charge and direction, it becomes eikonal.

- Eikonal lines

  Graphically, one may introduce eikonal Feynman rules

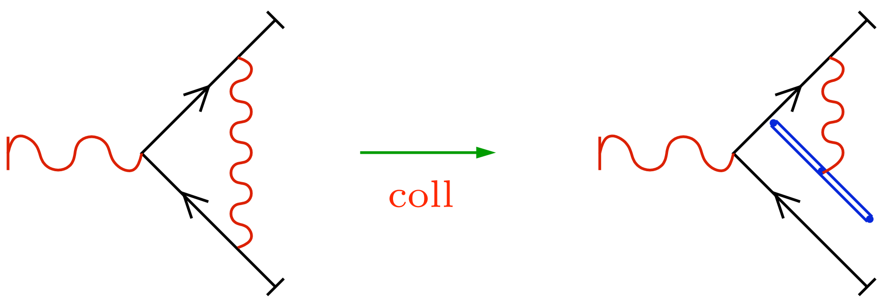


$= \mathrm{i} g u^{\alpha} t^{a}_{ij}$



$= \frac{\mathrm{i}\delta_{ij}}{u\cdot k + \mathrm{i}\varepsilon}$

  in terms of these rules the previous calculation reads



- More than one gluon

  Summing over all possible insertions of two or more collinear gluons, one finds sistematic cancellations, due to Ward identities. One uses then eikonal identities like

$$\frac{1}{k_1 \cdot u}\,\frac{1}{k_2 \cdot u} = \frac{1}{(k_1 + k_2)\cdot u}\,\frac{1}{k_1 \cdot u} + \frac{1}{(k_1 + k_2)\cdot u}\,\frac{1}{k_2 \cdot u}\,.$$

  All collinear gluons couple to the same eikonal line.

- Typical result

  In an axial gauge the form factor factorizes

$$\Gamma\left(\frac{q^2}{\mu^2}\right) = J_1\left(\frac{(p_1 \cdot n)^2}{\mu^2 n^2}\right) J_2\left(\frac{(p_2 \cdot n)^2}{\mu^2 n^2}\right) \mathcal{S}\left(u_i \cdot n\right) H\left(\frac{q^2}{\mu^2}\right)\,.$$

Multi-scale problems and large logarithms

For observables depending on a single hard scale all logarithms are resummed using the renormalization group

$$\sigma\left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \sigma\left(1, \alpha_s(q^2)\right) \ .$$

Most problems however have several hard scales. If for example $q_1^2 \gg q_2^2 \gg \Lambda_{\rm QCD}$, the reliability of perturbation theory is put in jeopardy by terms like $\alpha_s^n \log^p(q_1^2/q_2^2)$, with $p \leq n$ (single logarithms) or $p \leq 2n$ (double logarithms). They all arise from IR/C dynamics. Some examples.

- The "Sudakov" form factor.

$$\Gamma\left(\frac{q^2}{\mu^2}\right) = 1 - \frac{\alpha_s}{4\pi} C_F \log^2\left(\frac{q^2}{\mu^2}\right) + \ldots \ .$$

  In a massless theory one may choose $\mu^2 = q^2$, and be left with IR/C poles. With masses one finds duble logarithms $\log^2\left(q^2/m^2\right)$.

- DIS. The two hard scales are $Q^2 = -q^2$ and $W^2 = (p + q)^2 = Q^2(1-x)/x$. There are single $\log(1/x)$, resummed by the BFKL equation, and double $\log(1-x)$, resummed *à la* Sudakov.

- Drell–Yan. The process is $q\bar{q} \to \mu^+\mu^-(q^2)$, the two hard scales are $s = (p_1 + p_2)^2$ and q^2. Double $\log(1 - q^2/s)$ are resummed *à la* Sudakov.

- The transverse momentum distribution in Drell–Yan. The two scales are q^2 and $q_\perp^2$, giving double $\log(q_\perp^2/q^2)$.

Factorization and resummation

The deep connection between factorization and resummation, can already be seen from the renormalization group.

$$G_0^{(n)}(p_i, \Lambda, g_0) = \prod_{i=1}^{n} Z_i^{1/2}(\Lambda/\mu, g(\mu)) \; G_R^{(n)}(p_i, \mu, g(\mu)) \; ,$$

$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d \log G_R^{(n)}}{d \log \mu} = -\sum_{i=1}^{n} \gamma_i (g(\mu)) \; .$$

Solving this equation resums in exponential form the logarithmic dependence on μ.

The Altarelli–Parisi equation similarly leds to the resummation of (single) logarithms of μ_F. In terms of Mellin moments

$$\widetilde{F}_2\left(N, \frac{Q^2}{m^2}, \alpha_s(Q^2)\right) = \widetilde{C}\left(N, \frac{Q^2}{\mu_F^2}, \alpha_s(Q^2)\right) \widetilde{f}\left(N, \frac{\mu_F^2}{m^2}, \alpha_s(Q^2)\right) \; ,$$

$$\frac{d\widetilde{F}_2}{d\mu_F} = 0 \quad \rightarrow \quad \frac{d \log \widetilde{f}}{d \log \mu_F} = \gamma_N \left(\alpha_s(Q^2)\right) \; .$$

Double logarithms are more difficult. Ordinary renormalization group is not sufficient. Gauge invariance plays a key role. Or: use effective filed theory (SCET).

The quark form factor

Consider the factorization

$$\Gamma\left(\frac{q^2}{\mu^2}\right) = J_1\left(\frac{(p_1 \cdot n_1)^2}{\mu^2 n_1^2}\right) J_2\left(\frac{(p_2 \cdot n_2)^2}{\mu^2 n_2^2}\right) \mathcal{S}\left(u_i \cdot n_i\right) H\left(\frac{q^2}{\mu^2}, n_i\right) .$$

The form factor is gauge invariant, so that

$$\frac{\partial \log \Gamma}{\partial p_1 \cdot n_1} = 0 \;\; \rightarrow \;\; \frac{\partial \log J_1}{\partial \log(p_1 \cdot n_1)} = -\frac{\partial \log H}{\partial \log(p_1 \cdot n_1)} - \frac{\partial \log \mathcal{S}}{\partial \log(u_1 \cdot n_1)} \, .$$

The two functions on the $r.h.s.$ have different arguments. Then

$$\frac{\partial \log J}{\partial \log q} = K_J \left(\alpha_s(\mu^2), \epsilon \right) + G_J \left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \, .$$

The K function contains all singularities in ϵ (H is finite as $\epsilon \to 0$). The function G contains all q^2 dependence.

The whole form factor obeys an equation of identical form. Furthermore the form factor is renormalization group invariant, so that

$$\frac{dG}{d \log \mu} = -\frac{dK}{d \log \mu} = \gamma_K(\alpha_s(\mu)) \, ,$$

with a finite anomalous dimension, independent of q^2.

The equation for Γ can be solved. Since Γ is divergent, one needs to keep consistently the dependence on $\epsilon < 0$ to all orders. $\alpha_s(\mu^2)$ becomes $\alpha_s(\mu^2, \epsilon)$ which implies $\Gamma(q^2 = 0, \epsilon < 0) = 0$; then

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[K\left(\epsilon, \alpha_s(\mu^2)\right) + G\left(-1, \overline{\alpha}\left(\frac{\xi^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right), \epsilon\right) + \frac{1}{2} \int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K\left(\overline{\alpha}\left(\frac{\lambda^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right)\right) \right] \right\} .$$

The exponentiation is non trivial, the exponent has only single poles in ϵ of the form $\alpha_s^n / \epsilon^{n+1}$.

The thrust distribution

The thrust distribution is singular as $T \to 1$, behaving as $\alpha_s^n \log^{2n-1}(1-T)/(1-T)$. These logs can be resummed with similar methods.

As $T \to 1$ the distribution can be factorized in a manner similar to Γ. The jets J now enter the final state, so they have a non-vanishing invariant mass $m_J^2 \propto (1-T)q^2$ as $T \to 1$.

$$\sigma(N) \equiv \frac{1}{\sigma_0} \int_0^1 dT \, T^N \, \frac{d\sigma}{dT} = \hat{J}_1 \left(\frac{q^2}{N\mu^2}, \frac{(p_1 \cdot n)^2}{n^2 \mu^2} \right) \hat{J}_2 \, \hat{\mathcal{S}} \, \hat{H} \, .$$

In an axial gauge, leading logarithms are in the jet functions J, which satisfy

$$\frac{\partial \log \hat{J}}{\partial \log q} = \hat{K}_J \left(\frac{q^2}{N\mu^2}, \alpha_s(\mu^2) \right) + \hat{G}_J \left(\frac{q^2}{\mu^2}, \alpha_s(\mu^2) \right) \, .$$

One can then solve for $\hat{J}(N)$ in terms of $\hat{J}(1)$, as

$$\hat{J} \left(\frac{q^2}{N\mu^2} \right) = \hat{J} \left(\frac{q^2}{\mu^2} \right) \exp \left[-\frac{1}{2} \int_{q^2/N}^{q^2} \frac{d\lambda^2}{\lambda^2} \left(\log \frac{\mu}{\lambda} \Gamma_{\hat{J}} \left(\alpha_s(\lambda^2) \right) - \Gamma'_{\hat{J}} \left(\alpha_s(\lambda^2) \right) \right) \right] .$$

Leading logarithms are determined by $\Gamma_{\hat{J}} = \gamma_K + \ldots$. Neglecting running coupling effects one easily finds $\hat{J} \sim \exp(\alpha_s \log^2 N)$. Inverting the Mellin transform,

$$\frac{1}{\sigma_0} \frac{d\sigma}{dT} = -2C_F \frac{\alpha_s}{\pi} \frac{\log(1-T)}{1-T} \exp \left[-C_F \frac{\alpha_s}{\pi} \log^2(1-T) \right] \, .$$

Note: $d\sigma/dT \to 0$ as $T \to 1$ ("Sudakov suppression").

The borders of the perturbative regime

Resummations test the limits of the perturbative theory. One gets in fact integrals of the form

$$f_a(q^2) = \int_0^{q^2} \frac{dk^2}{k^2} (k^2)^a \alpha_s(k^2) \; ,$$

One sees esplicitly the non–convergence of perturbation theory at high orders, consequence of the Landau pole. In fact

$$\alpha_s(k^2) = \frac{\alpha_s(q^2)}{1 + \beta_0 \alpha_s(q^2) \log\left(k^2/q^2\right)} \; .$$

Letting $z \equiv \log q^2/k^2$, one observes that

- The perturbative expansion in powers of $\alpha_s(q^2)$ diverges.

$$f_a(q^2) = (q^2)^a \sum_{n=0}^{\infty} \beta_0^n \alpha_s^{n+1}(q^2) \int_0^{\infty} dz \mathrm{e}^{-az} z^n = \sum_{n=1}^{\infty} c_n \alpha_s^n n! \; .$$

- The integral is ambiguous, but the ambiguity is suppressed by powers of q^2. In fact

$$f_a(q^2) = (q^2)^a \alpha_s(q^2) \int_0^{\infty} dz \frac{\mathrm{e}^{-az}}{1 - \beta_0 \alpha_s(q^2) z} \; .$$

  The Landau pole on the integrazion contour induces an ambiguity of the same parametric size as the residue

$$\left| \delta f_a(q^2) \right| \propto \exp\left[- \frac{a}{\beta_0 \alpha_s(q^2)} \right] = \left(\frac{\Lambda_{\rm QCD}^2}{q^2} \right)^a \; .$$