Exponentiation at partonic threshold for the Drell–Yan cross section

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Abstract

The techniques leading to the resummation of threshold logarithms in the Drell–Yan cross section and other processes can be used to show that also terms independent on the Mellin variable N exponentiate. Comparison with explicit two–loop calculations shows that within this class of terms the exponentiation of the one–loop result together with the running of the coupling is the dominant effect at two loops.

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Summary

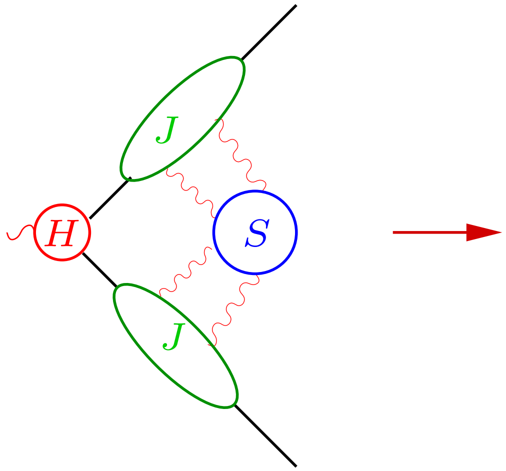
- Tools: factorizations and resummations
  - The Sudakov form factor $\Gamma(Q^2, \epsilon)$
  - The unsubtracted Drell–Yan cross section $\omega(N, Q^2, \epsilon)$
  - The unsubtracted DIS cross section $F_2(N, Q^2, \epsilon)$
  - The $\overline{\mathrm{MS}}$ quark distribution $\phi_{\overline{\mathrm{MS}}}(N, Q^2, \epsilon)$
- DIS factorization scheme
  - Virtual diagrams: recovering the ratio of form factors
  - Real diagrams: running couplings
  - Complete exponentiation
- $\overline{\mathrm{MS}}$ factorization scheme
  - Virtual diagrams: cancelling virtual poles
  - Real diagrams: running couplings
  - Complete exponentiation
- Does it work?
  - Analytical tests
  - Numerical tests
- Outlook – future work

Factorization of $\Gamma(Q^2, \epsilon)$

Begin by considering the quark form factor – an important source of N-independent terms

$$\Gamma_\mu(q, \epsilon) = \langle p_1, p_2 | J_\mu(0) | 0 \rangle = -\mathrm{i} e e_q \, \overline{u}(p_1) \gamma_\mu v(p_2) \, \Gamma\left(Q^2, \epsilon\right) \ .$$

In dimensional regularization the form factor obeys a factorization of the form



$$\Gamma\left(Q^2, \epsilon\right) = J\left(p_1 \cdot n, \epsilon\right) J\left(p_2 \cdot n, \epsilon\right) S\left(u_i \cdot n, \epsilon\right) H\left(Q^2\right) \ .$$

Every factorization leads to an evolution equation. In this case

$$Q^2 \frac{\partial}{\partial Q^2} \log\left[\Gamma\left(Q^2, \epsilon\right)\right] = \frac{1}{2}\left[K\left(\epsilon\right) + G\left(Q^2, \epsilon\right)\right] \ .$$

The evolution equation can be solved in $d = 4 - 2\epsilon$, with the boundary condition $\Gamma(0, \epsilon < 0) = 1$, in terms of the d-dimensional running coupling

$$\overline{\alpha}\left(\xi^2\right) = \alpha_s(\mu^2)\left[\left(\frac{\xi^2}{\mu^2}\right)^{\epsilon} - \frac{1}{\epsilon}\left(1 - \left(\frac{\xi^2}{\mu^2}\right)^{\epsilon}\right)\frac{b_0}{4\pi}\alpha_s(\mu^2)\right]^{-1} .$$

The result is an exponential of functions of only the running coupling

$$\Gamma\left(Q^2, \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{-Q^2}\frac{d\xi^2}{\xi^2}\left[K\left(\epsilon\right) + G\left(\overline{\alpha}\left(\xi^2\right), \epsilon\right) + \frac{1}{2}\int_{\xi^2}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\overline{\alpha}\left(\lambda^2\right)\right)\right]\right\} .$$

Here $\gamma_K(\alpha_s)$ is the cusp anomalous dimension of the Wilson line representing the quarks.

This resummation of IR and collinear poles leads to a simple representation for the ratio of timelike to spacelike form factors, in terms of a contour integral in the complex Q^2 plane.

$$\left|\frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)}\right| = \exp\left\{\left|\frac{\mathrm{i}}{2}\int_0^{\pi}\left[G\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\theta}Q^2\right), \epsilon\right) - \frac{\mathrm{i}}{2}\int_0^{\theta}d\phi\,\gamma_K\left(\overline{\alpha}\left(\mathrm{e}^{\mathrm{i}\phi}Q^2\right)\right)\right]\right|\right\} .$$

which is manifestly finite as $\epsilon \to 0$.

Expanding the running couplings in terms of $\alpha_s(Q^2)$ one predicts that at two loops

$$\left|\frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)}\right|^2 = 1 + \frac{\alpha_s}{\pi}\frac{3\zeta_2\gamma_K^{(1)}}{2} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{9}{8}\zeta_2^2\left(\gamma_K^{(1)}\right)^2 + \frac{3}{4}\zeta_2 b_0 G^{(1)}(0) + \frac{3}{2}\zeta_2\gamma_K^{(2)}\right] ,$$

where

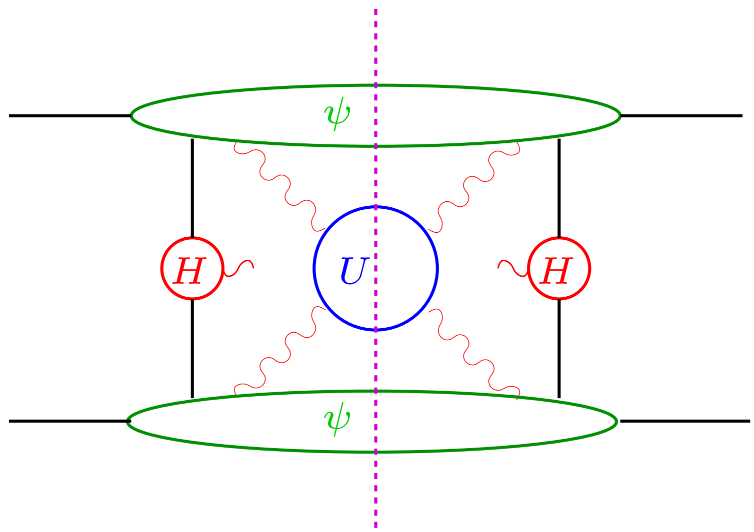
$$\gamma_K^{(1)} = 2C_F \quad ; \quad G^{(1)}(0) = \frac{3}{2}C_F \quad ; \quad \gamma_K^{(2)} = C_A C_F\left(\frac{67}{18} - \zeta_2\right) - n_f C_F\left(\frac{5}{9}\right) .$$

This illustrates the "predictive power" of exponentiation, even when applied to coefficients with no dependence on dynamical scales.

- The finiteness of the ratio can be predicted to all orders.
- Numerically the exponentiation of the one loop result together with running coupling effects provides roughly three quarters of the two–loop result.
- The only genuine two–loop contribution to this quantity is given by the cusp anomalous dimension, much simpler to compute than the full form factor.

Factorization of $\omega(N, Q^2, \epsilon)$

The resummation of threshold logarithms in the Drell–Yan process ($\log(Q^2/s) \to \log N$ upon Mellin transform) relies upon a factorization of the unsubtracted cross section ω at large N.



In an axial gauge ψ is a parton distribution containing collinear and soft-collinear enhancements, U is an eikonal function responsible for wide-angle soft emission. One finds

$$\omega(N, \epsilon) = |H_{\rm DY}|^2 \, \psi(N, \epsilon)^2 \, U(N) + \mathcal{O}(1/N) \, .$$

To recover the form factor, one must separate purely virtual contribution from real emission.

$$\begin{aligned} \psi(N, \epsilon) &= \mathcal{R}(\epsilon) \, \psi_R(N, \epsilon) \, , \\ U(N) &= U_V(\epsilon) \, U_R(N, \epsilon) \, , \end{aligned}$$

where $\mathcal{R}(\epsilon)$ is the residue of the axial gauge quark propagator.

The result is

$$\omega(N,\epsilon) = |H_{\rm DY}\,\mathcal{R}(\epsilon)\sqrt{U_V(\epsilon)}|^2\,\psi_R(N,\epsilon)^2\,U_R(N,\epsilon) + \mathcal{O}(1/N)$$
$$= |\Gamma(Q^2,\epsilon)|^2\,\psi_R(N,\epsilon)^2\,U_R(N,\epsilon) + \mathcal{O}(1/N)\ .$$

Purely virtual contributions reconstruct the form factor squared.

Real emission contributions (at least one gluon in the final state) can also be shown to exponentiate (Sterman 1987). Up to $1/N$ corrections

$$\psi_R(N,\epsilon) = \exp\left[\int_0^1 dz \frac{z^{N-1}}{1-z}\int_z^1 \frac{dy}{1-y}\kappa_\psi\Big(\overline{\alpha}\left((1-z)^2Q^2\right),\epsilon\Big)\right]\ .$$

Eikonal graphs also exponentiate, thus

$$U_R(N,\epsilon) = \exp\left[-\int_0^1 dz\,\frac{z^{N-1}}{1-z}\,g_U\left(\overline{\alpha}\left((1-z)^2Q^2\right),\epsilon\right)\right]\ .$$

The functions κ_ψ and g_U can be explicitly defined as operator matrix elements. At one loop

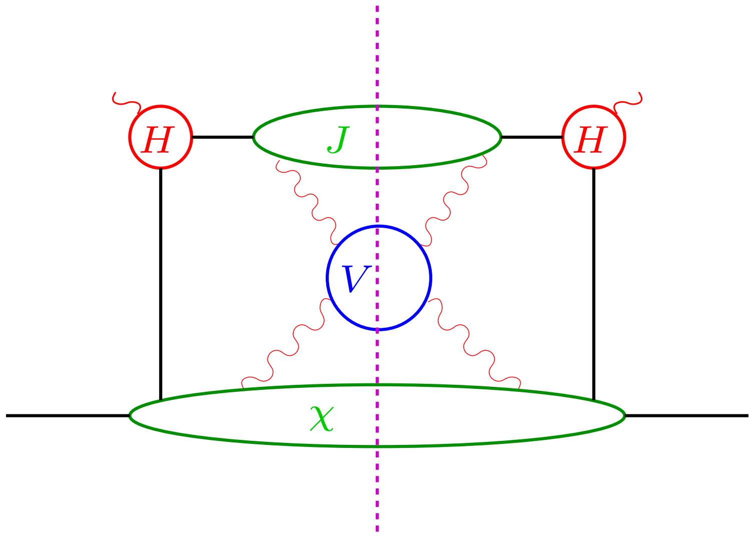
$$\kappa_\psi^{(1)} = 2C_F\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\quad ;\quad g_U^{(1)} = -2C_F\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\ .$$

Infrared and collinear poles are explicitly generated by the integrations.

We can conclude that $\omega(N,\epsilon)$ exponentiates up to corrections suppressed by powers of $1/N$.

Factorization of $F_2(N, Q^2, \epsilon)$

A similar factorization can be performed at large N ($x_{Bj} \to 1$) on the unsubtracted DIS structure function F_2.



The functions χ and V differ from their counterparts ψ and U because of their phase space integration. The function J represent the narrow current jet (of mass $(1-x)Q^2$). One finds

$$F_2(N, Q^2, \epsilon) = |H_{\rm DIS}|^2 \, \chi(N, \epsilon) \, V(N) \, J(N) + \mathcal{O}(1/N) \, .$$

Separating again virtual contributions from real emission, and noting that J_V is again given by the residue of the quark propagator, F_2 can be expressed as

$$F_2(N, \epsilon) = |\Gamma(-Q^2, \epsilon)|^2 \, \chi_R(N, \epsilon) \, V_R(N, \epsilon) \, J_R(N, \epsilon) \, .$$

Once again purely virtual contributions reconstruct the form factor squared. Real emission contributions exponentiate as they do in the case of ω.

The $\overline{\mathrm{MS}}$ quark distribution $\phi_{\overline{\mathrm{MS}}}(N, Q^2, \epsilon)$

The $\overline{\mathrm{MS}}$ quark distribution is known to exponentiate near threshold. One can write

$$\phi_{\overline{\mathrm{MS}}}\left(N, Q^2, \epsilon\right) = \exp\left[\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left(B_\delta\left(\overline{\alpha}(\xi^2)\right)\right.\right.$$

$$\left.\left. + \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} A\left(\overline{\alpha}(\xi^2)\right)\right)\right] + \mathcal{O}(1/N)\,.$$

The function A is essentially the cusp anomalous dimension, the function B_δ is the virtual part of the nonsinglet quark splitting function.

$$A^{(1)} = C_F \quad ; \quad A^{(2)} = \frac{1}{2}\gamma_K^{(2)} \quad ; \quad B_\delta^{(1)} = \frac{3}{4} C_F\,.$$

It is useful to split the distribution into "virtual" and "real" contributions, matching the corresponding factorization of $\omega(N, Q^2, \epsilon)$. Define then ϕ_V to cancel all poles of the timelike form factor, by

$$\phi_V(Q^2, \epsilon) = \exp\left\{\frac{1}{2}\int_0^{Q^2} \frac{d\xi^2}{\xi^2}\left[K(\epsilon) + \widetilde{G}\left(\overline{\alpha}(\xi^2)\right)\right.\right.$$

$$\left.\left. + \frac{1}{2}\int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2}\,\gamma_K\left(\overline{\alpha}(\lambda^2)\right)\right]\right\}\,,$$

with the same counterterm K and anomalous dimension γ_K as the form factor, but a different finite correction $\widetilde{G}$.

$\widetilde{G}(\alpha_s)$ is uniquely determined by the following criteria

- It must be finite and independent of ϵ, not to generate finite correction to the $\overline{\mathrm{MS}}$ distribution, which must have only poles.
- The perturbative coefficients of $\widetilde{G}$ must be chosen to cancel all poles in the (real) ratio $\Gamma(-Q^2, \epsilon)/\phi_V(Q^2, \epsilon)$.

It can be shown recursively that such a $\widetilde{G}$ exists, and its perturbative coefficients can be computed in terms of those of G. Let

$$G\left(\alpha_s, \epsilon\right) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} G_n^{(m)} \epsilon^m \left(\frac{\alpha_s}{\pi}\right)^n .$$

then one finds

$$\begin{aligned} \widetilde{G}_{M+1} &= G^{(0)}_{M+1} - \frac{b_0}{4} G^{(1)}_{M} - \frac{b_1}{4} G^{(1)}_{M-1} + \frac{b_0^2}{16} G^{(2)}_{M-1} - \frac{b_2}{4} G^{(1)}_{M-2} \\ &+ \frac{b_0 b_1}{8} G^{(2)}_{M-2} - \frac{b_0^3}{64} G^{(3)}_{M-2} + \ldots \end{aligned}$$

The real part of $\phi_{\overline{\mathrm{MS}}}$ is then defined by subtraction, as $\phi_R = \phi_{\overline{\mathrm{MS}}}/\phi_V$. Note that ϕ_V and ϕ_R have cancelling double poles of IR-collinear nature, while $\phi_{\overline{\mathrm{MS}}}$ has only collinear simple poles.

Exponentiation in the DIS scheme

Collecting the results for the factorizations of ω and F_2 one can construct the Drell–Yan hard part in the DIS scheme as

$$\widehat{\omega}_{\rm DIS}(N) = \frac{1}{|H_{DIS}|^2} \left| \frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)} \right|^2 \frac{\psi_R(N,\epsilon)}{\chi_R(N,\epsilon)} \frac{U(N)}{V^2(N)} \frac{1}{J^2(N)} \, ,$$

One recognizes the ratio of form factors (Parisi, 1980). Collecting all other exponentiated terms in a familiar form one finds

$$\begin{aligned} \widehat{\omega}_{\rm DIS}(N) \; &= \; \left| \frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)} \right|^2 \exp \left[F_{DIS}(N,\alpha_s) \right] \\ &\times \; \exp \left[\int_0^1 dz \, \frac{z^{N-1}-1}{1-z} \left\{ 2 \int_{(1-z)Q^2}^{(1-z)^2 Q^2} \frac{d\xi^2}{\xi^2} A\left(\alpha_s(\xi^2)\right) \right. \right. \\ &\left. \left. - \; 2B\left(\alpha_s\left((1-z)Q^2\right)\right) + D\left(\alpha_s\left((1-z)^2 Q^2\right)\right) \; \right\} \right] . \end{aligned}$$

where

$$F_{DIS}^{(1)} = C_F \left(\frac{1}{2} + \zeta_2 \right) \quad ; \quad D^{(1)} = 0 \qquad (1)$$

while a subset of terms in $F_{DIS}^{(2)}$ are dictated by running coupling

$$\begin{aligned} F_{DIS}^{(2)} \quad &= \quad \frac{1}{2} C_F b_0 \Big[(2+\zeta_2)\left(\log N + \gamma_E\right) \\ &- \quad \frac{3}{8}\left(4 + \zeta_2 - 2\zeta_3\right) \Big] + \delta F_{DIS}^{(2)} . \end{aligned}$$

Exponentiation in the $\overline{\rm MS}$ scheme

For the Drell–Yan hard part in the $\overline{\rm MS}$ scheme we have

$$\widehat{\omega}_{\overline{\rm MS}}(N) = \left| \frac{\Gamma(Q^2,\epsilon)}{\phi_V(Q^2,\epsilon)} \right|^2 \left[U_R(N,\epsilon) \left(\frac{\psi_R(N,\epsilon)}{\phi_R(N,\epsilon)} \right)^2 \right] ,$$

This also can be expressed as a product of real and finite terms, in the familiar form

$$\widehat{\omega}_{\overline{\rm MS}}(N) = \left| \frac{\Gamma(Q^2,\epsilon)}{\Gamma(-Q^2,\epsilon)} \right|^2 \cdot \left(\frac{\Gamma(-Q^2,\epsilon)}{\phi_V(Q^2,\epsilon)} \right)^2 \cdot \exp\left[F_{\overline{\rm MS}}(\alpha_s) \right]$$
$$\times \exp\left[\int_0^1 dz \, \frac{z^{N-1}-1}{1-z} \left\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A\left(\alpha_s(\mu^2)\right) + D\left(\alpha_s\left((1-z)^2 Q^2\right)\right) \right\} \right] .$$

where D is the same function appearing in the DIS scheme, and

$$F^{(1)}_{\overline{\rm MS}} = -\frac{3}{2} C_F \zeta_2 ,$$
$$F^{(2)}_{\overline{\rm MS}} = -\frac{1}{4} C_F b_0 \left(1 - \frac{3}{8}\zeta_2 + \frac{7}{16}\zeta_3 \right) + \delta F^{(2)}_{\overline{\rm MS}} ,$$

Note that the full set of constants at two loops can be determined only with a two–loop calculation, although simpler than the full cross section.

Does it work?

Exponentiation techniques described here can be applied/tested in at least two different ways

- Analytical results: factorization into real and virtual parts, separately finite, leads to simplified calculations.
  - RG invariance of the exponent κ_ψ of the distribution ψ leads to

$$\kappa_\psi(\xi,\epsilon) = \frac{\alpha_s}{\pi}\kappa_\psi^{(1)}(\epsilon)\ \xi^{-2\epsilon}$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \left(\kappa_\psi^{(2)}(\epsilon)\ \xi^{-4\epsilon} + \frac{b_0}{4\epsilon}\kappa_\psi^{(1)}(\epsilon)\ \xi^{-2\epsilon}\left(\xi^{-2\epsilon}-1\right)\right)\ ,$$

  - Let $\kappa_\psi^{(2)}(\epsilon) \equiv \sum_{m=0}^{\infty} \kappa_{\psi,m}^{(2)}\ \epsilon^m$. One can show that
  - NNL logs at two loops are completely determined by $\kappa_{\psi,0}^{(2)}$
  - $\kappa_{\psi,0}^{(2)}$ is in turn determined by the cancellation of double poles in the ratio $U_R\psi_R/\phi_R$ as

$$\kappa_{\psi,0}^{(2)} = \gamma_K^{(2)} + \frac{1}{2}C_F b_0\ .$$

  - Similarly, cancellation of simple poles in the ratio $U_R\psi_R/\phi_R$ determines the function $D(\alpha_s)$ at two loops in terms of the purely virtual quantities $G^{(2)}(\epsilon=0)$ and $B_\delta^{(2)}$, as

$$D^{(2)} = \frac{3}{4}C_F b_0 \zeta_2 + 4B_\delta^{(2)} - 2\widetilde{G}^{(2)}\ .$$

- Numerical impact

  Given a g–loop calculation of one of the cross sections discussed, one can use exponentiation and RG to estimate the $(g+1)$–loop result. Given the existing results at two loops (Van Neerven *et al.*, 1992), one can test the case $g=1$. Define

$$\omega_s(N) = \sum_{p=0}^{\infty} \omega_s^{(p)}(N) \left(\frac{\alpha_s}{\pi}\right)^p ,$$

$$\omega_s^{(p)}(N) = \sum_{i=0}^{2p} \omega_{s,i}^{(p)} \left(\log N + \gamma_E\right)^i ,$$

$$\Delta\omega_{s,i}^{(2)} \equiv \frac{\widetilde{\omega}_{s,i}^{(2)} - \omega_{s,i}^{(2)}}{\omega_{s,i}^{(2)}} .$$

  where $\widetilde{\omega}_{s,i}^{(2)}$ is the estimate obtained using only one loop results with running coupling effects. One finds

i	4	3	2	1	0
$\overline{\mathrm{MS}}$	0	0	- 0.33	- 1.79	0.69
DIS	0	0	- 0.13	- 1.17	- 0.26

  – Satisfactory result for constants, particularly DIS scheme.
  – Overcompensated running coupling effects for $i = 1$. "Excess" of factorization?
  – Discrepancy at $i=2$ disappears introducing $\gamma_K^{(2)}$.

Outlook

- We have shown that threshold resummation implies complete exponentiation of N–independent terms for the Drell–Yan cross section, both in the DIS and $\overline{\mathrm{MS}}$ scheme.

  Corollary: the same applies to the $\overline{\mathrm{MS}}$ scheme DIS cross section.
- We have tested the exponentiation technique reproducing known results and gauged the numerical impact at two loops. Running coupling effects deteriorate the prediction at single log level, while constants are correctly estimated.
- Exponentiation of of N–independent terms does not have the same predictive power as for threshold logarithms. It can be used to gauge impact of higher order corrections and simplify computation of given subsets of them.
- What else exponentiates? Empirical evidence suggests $\log(N)/N$ terms do in the Drell–Yan cross section.

  Note: $\log(N)/N$ terms are known to have considerable numerical impact. Worth studying.
- Other processes? Threshold resummation applies to cross sections with multiple colored particles ($e.g.$ $t\bar{t}$ production). Do "large π^2" terms exponentiate also in these cases?